Tutorial sheet 1

Discussion topic: Which idealizations underlie the description of a macroscopic many-body system as a continuous medium? How is local thermodynamic equilibrium defined?

1. Wave equation

Consider a scalar field $\phi(t,x)$ which obeys the partial differential equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(t,x) = 0 \tag{1}$$

with initial conditions $\phi(0,x) = \mathrm{e}^{-x^2}$, $\partial_t\phi(0,x) = 0$. Determine the solution $\phi(t,x)$ for $t > 0$.

2. Stationary flow: first example

(This exercise and the following one make use of concepts which will only be seen in the lecture on April 14 or 16; this should pose you no difficulty.)

Consider the stationary flow defined in the region $x^1 > 0$, $x^2 > 0$ by its velocity field

$$\vec{\mathsf{v}}(t,\vec{r}) = k(-x^1\,\vec{\mathrm{e}}_1 + x^2\,\vec{\mathrm{e}}_2) \tag{2}$$

with k a positive constant, $\{\vec{\mathrm{e}}_i\}$ the basis vectors of a coordinate system and $\{x^i\}$ the coordinates of the position vector $\vec{r}$.

Determine the *stream lines* at some arbitrary time t. The latter are by definition lines $\vec{\xi}(\lambda)$ whose tangent is everywhere parallel to the instantaneous velocity field, with λ a parameter along the stream line. That is, they obey the condition

$$\frac{\mathrm{d}\vec{\xi}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\,\vec{\mathsf{v}}(t,\vec{\xi}(\lambda))$$

with $\alpha(\lambda)$ a scalar function, or equivalently

$$\frac{\mathrm{d}\xi^1(\lambda)}{\mathsf{v}^1(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^2(\lambda)}{\mathsf{v}^2(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^3(\lambda)}{\mathsf{v}^3(t,\vec{\xi}(\lambda))},$$

with $\mathrm{d}\xi^i(\lambda)$ the coordinates of the (infinitesimal) tangent vector to the stream line.

3. Stationary flow: second example

Consider the fluid flow whose velocity field $\vec{\mathsf{v}}(t,\vec{r})$ has coordinates (in a given Cartesian system)

$$\mathsf{v}^1(t,\vec{r}) = kx^2, \quad \mathsf{v}^2(t,\vec{r}) = kx^1, \quad \mathsf{v}^3(t,\vec{r}) = 0, \tag{3}$$

where k is a positive real number, while x^1, x^2, x^3 are the coordinates of the position vector $\vec{r}$.

i. Determine the stream lines (see definition in exercise **2.**) at an arbitrary instant t.

ii. Let X^1, X^2, X^3 denote the coordinates of some arbitrary point M and let t_0 be the real number defined by

$$kt_0 = \begin{cases} -\mathrm{Artanh}(X^2/X^1) & \text{if } |X^1| > |X^2| \\ 0 & \text{if } X^1 = \pm X^2 \\ -\mathrm{Artanh}(X^1/X^2) & \text{if } |X^1| < |X^2|. \end{cases}$$

Write down a parameterization $x^1(t)$, $x^2(t)$, $x^3(t)$, in terms of a parameter denoted by t, of the coordinates of *the* stream line $\vec{x}(t)$ going through M such that $\mathrm{d}\vec{x}(t)/\mathrm{d}t$ at any point *equals* the velocity field at that point, and that either $x^1(t) = 0$ or $x^2(t) = 0$ for $t = t_0$.

iii. Viewing $\vec{x}(t)$ as the trajectory of a point—actually, of a fluid particle—, you already know the velocity of that point at time t (do you?). What is its acceleration $\vec{a}(t)$?

iv. Coming back to the velocity field (3), compute first its partial derivative $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$, then the *material derivative*

$$\frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t} \equiv \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}),$$

where $\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}$ denotes the differential operator $\mathsf{v}^1(t,\vec{r})\partial_1 + \mathsf{v}^2(t,\vec{r})\partial_2 + \mathsf{v}^3(t,\vec{r})\partial_3$, with $\partial_i \equiv \partial_{x^i}$. Compare $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$ and $\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})/\mathrm{D}t$ with the acceleration of a fluid particle found in question **iii.**

Hint: You should review this exercise after the lectures on *Lagrangian* and *Eulerian descriptions*, even if you have had no problem in solving it earlier.