Tutorial sheet 1

Discussion topic: Which idealizations underlie the description of a macroscopic many-body system as a continuous medium? How is local thermodynamic equilibrium defined?

1. Wave equation

Consider a scalar field $\phi(t,x)$ which obeys the partial differential equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(t,x) = 0 \tag{1}$$

with initial conditions $\phi(0,x) = \mathrm{e}^{-x^2}$, $\partial_t\phi(0,x) = 0$. Determine the solution $\phi(t,x)$ for $t > 0$.

2. Stationary flow: first example

(This exercise and the following one make use of concepts which will only be seen in the lecture on April 14 or 16; this should pose you no difficulty.)

Consider the stationary flow defined in the region $x^1 > 0$, $x^2 > 0$ by its velocity field

$$\vec{\mathsf{v}}(t,\vec{r}) = k(-x^1\,\vec{\mathrm{e}}_1 + x^2\,\vec{\mathrm{e}}_2) \tag{2}$$

with k a positive constant, $\{\vec{\mathrm{e}}_i\}$ the basis vectors of a coordinate system and $\{x^i\}$ the coordinates of the position vector $\vec{r}$.

Determine the *stream lines* at some arbitrary time t. The latter are by definition lines $\vec{\xi}(\lambda)$ whose tangent is everywhere parallel to the instantaneous velocity field, with λ a parameter along the stream line. That is, they obey the condition

$$\frac{\mathrm{d}\vec{\xi}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\,\vec{\mathsf{v}}(t,\vec{\xi}(\lambda))$$

with $\alpha(\lambda)$ a scalar function, or equivalently

$$\frac{\mathrm{d}\xi^1(\lambda)}{\mathsf{v}^1(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^2(\lambda)}{\mathsf{v}^2(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^3(\lambda)}{\mathsf{v}^3(t,\vec{\xi}(\lambda))},$$

with $\mathrm{d}\xi^i(\lambda)$ the coordinates of the (infinitesimal) tangent vector to the stream line.

3. Stationary flow: second example

Consider the fluid flow whose velocity field $\vec{\mathsf{v}}(t,\vec{r})$ has coordinates (in a given Cartesian system)

$$\mathsf{v}^1(t,\vec{r}) = kx^2,\quad \mathsf{v}^2(t,\vec{r}) = kx^1,\quad \mathsf{v}^3(t,\vec{r}) = 0, \tag{3}$$

where k is a positive real number, while x^1, x^2, x^3 are the coordinates of the position vector $\vec{r}$.

i. Determine the stream lines (see definition in exercise **2.**) at an arbitrary instant t.

ii. Let X^1, X^2, X^3 denote the coordinates of some arbitrary point M and let t_0 be the real number defined by

$$kt_0 = \begin{cases} -\mathrm{Artanh}(X^2/X^1) & \text{if } |X^1| > |X^2| \\ 0 & \text{if } X^1 = \pm X^2 \\ -\mathrm{Artanh}(X^1/X^2) & \text{if } |X^1| < |X^2|. \end{cases}$$

Write down a parameterization $x^1(t)$, $x^2(t)$, $x^3(t)$, in terms of a parameter denoted by t, of the coordinates of *the* stream line $\vec{x}(t)$ going through M such that $\mathrm{d}\vec{x}(t)/\mathrm{d}t$ at any point *equals* the velocity field at that point, and that either $x^1(t) = 0$ or $x^2(t) = 0$ for $t = t_0$.

iii. Viewing $\vec{x}(t)$ as the trajectory of a point—actually, of a fluid particle—, you already know the velocity of that point at time t (do you?). What is its acceleration $\vec{a}(t)$?

iv. Coming back to the velocity field (3), compute first its partial derivative $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$, then the *material derivative*

$$\frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t} \equiv \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}),$$

where $\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}$ denotes the differential operator $\mathsf{v}^1(t,\vec{r})\partial_1 + \mathsf{v}^2(t,\vec{r})\partial_2 + \mathsf{v}^3(t,\vec{r})\partial_3$, with $\partial_i \equiv \partial_{x^i}$. Compare $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$ and $\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})/\mathrm{D}t$ with the acceleration of a fluid particle found in question **iii.**

Hint: You should review this exercise after the lectures on *Lagrangian* and *Eulerian descriptions*, even if you have had no problem in solving it earlier.

Tutorial sheet 2

Discussion topics: What are the Lagrangian and Eulerian descriptions? How is a fluid defined?

4. Lagrangian description: Jacobian determinant

Consider the twice continuously differentiable ($\mathscr{C}^2$) mapping $(t,\vec{R}) \mapsto \vec{r}(t,\vec{R})$ from "initial" position vectors at t_0 to those at time t. Let (X^1,X^2,X^3) resp. (x^1,x^2,x^3) denote the coordinates of $\vec{R}$ resp. $\vec{r}$ in some fixed system.

The *Jacobian determinant* $J(t,\vec{R})$ of the transformation $\vec{R} \mapsto \vec{r}$ is as usual the determinant of the matrix with elements $\partial x^i/\partial X^j$. Thanks to the hypotheses on the mapping $\vec{r}(t,\vec{R})$, this Jacobian has simple mathematical properties.

i. Can you find a physical interpretation for $J(t,\vec{R})$? [*Hint*: Think of small volume elements.]

ii. Using the initial value $J(t_0,\vec{R})$ in the reference configuration, as well as the invertibility and $\mathscr{C}^2$-character of the mapping $\vec{r}(t,\vec{R})$, show that $J(t,\vec{R})$ is positive for $t \geq t_0$. What does this mean physically?

iii. Consider the motion of a continuous medium defined for $t \geq 0$ by

$$x^1 = X^1 + ktX^2, \quad x^2 = X^2 + ktX^1, \quad x^3 = X^3,$$

where $k > 0$. One may for simplicity assume that the coordinates are Cartesian.

a) Over which time range is this motion defined? [*Hint*: Jacobian determinant!]

b) What are its pathlines?

c) Determine the Eulerian description of this motion, i.e. the velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

5. Stress tensor

Let $\mathbf{T}_{ij}$ denote the Cartesian[1] components of the stress tensor in a continuous medium. Consider an infinitesimal cube of medium of side $\mathrm{d}\ell$, whose sides are parallel to the axes of the coordinate system.

i. Explain why the k-component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the continuous medium obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\mathbf{T}_{ij}(\mathrm{d}\ell)^3$, with ϵ_{ijk} the usual Levi-Civita symbol.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the continuum mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?

6. Isotropy of pressure

Consider a geometrical point at position $\vec{r}$ in a fluid at rest. The stress vector across every surface element going through this point is normal: $\vec{T}(\vec{r}) = -\mathcal{P}(\vec{r})\,\vec{e}_n$, with $\vec{e}_n$ the unit vector orthogonal to the surface element under consideration. Show that the (hydrostatic) pressure $\mathcal{P}$ is independent of the orientation of $\vec{e}_n$.

Hint: Consider the forces on the faces of an infinitesimal trirectangular tetrahedron.

[1]... which allows us to be sloppy with the position of indices.

Tutorial sheet 3

Discussion topics:

- What are the strain rate tensor, the rotation rate tensor, and the vorticity vector? How do they come about and what do they measure?
- What is the Reynolds transport theorem (and its utility)?
(- Give the basic equations governing the dynamics of perfect fluids.)

7. Example of a motion

Consider the motion defined in a system of Cartesian coordinates with basis vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ by the velocity field with components

$$\mathsf{v}^1(t,\vec{r}) = f_1(t,x^2), \quad \mathsf{v}^2(t,\vec{r}) = f_2(t,x^1), \quad \mathsf{v}^3(t,\vec{r}) = 0,$$

with f_1, f_2 two continuously differentiable functions.

i. Compute the strain rate tensor $\mathbf{D}(t,\vec{r})$ for this motion. What are its principal axes and the corresponding eigenvalues?[1] What is the volume expansion rate?

ii. Give the rotation rate tensor $\mathbf{R}(t,\vec{r})$ and the vorticity vector. Under which condition(s) on the functions f_1, f_2 does the motion become irrotational?

8. Pointlike source

Consider the fluid motion defined in a system of cylindrical coordinates (r,θ,z) by the velocity field given for $r \neq 0$ by

$$\mathsf{v}^r(t,\vec{r}) = \frac{f(t)}{r}, \quad \mathsf{v}^\theta(t,\vec{r}) = 0, \quad \mathsf{v}^z(t,\vec{r}) = 0,$$

with f some scalar function.

i. Calculate the strain rate tensor; what are its principal axes? Give the volume expansion rate. Compute the vorticity vector.

ii. Mathematically, the velocity field is singular at $r = 0$. Thinking of the velocity profile, what do you have *physically* at that point if $f(t) > 0$? if $f(t) < 0$?

9. Pointlike vortex

Consider now the motion defined in a system of cylindrical coordinates by the velocity field given for $r \neq 0$ by

$$\vec{\mathsf{v}}(t,r,\theta,z) = \frac{\Gamma}{2\pi r}\vec{e}_\theta, \quad \Gamma \in \mathbb{R}.$$

Give the strain rate tensor, with its principal axes and eigenvalues, the volume expansion rate, the rotation rate tensor and the vorticity vector. Compute the *circulation* of the velocity field along a closed curve circling the z-axis.

[1] Need a reminder on these notions? Check your favorite lecture on the mechanics of rigid bodies, especially the chapter on the tensor of inertia: e.g. `http://www.physik.uni-bielefeld.de/~yorks/theo1/` (Nov.11 & 12 lectures) or `http://www.physik.uni-bielefeld.de/~laine/klassisch/` (Nov.12 lecture).

Tutorial sheet 4

Discussion topics:

- Give the basic equations governing the dynamics of perfect fluids.
- What is the Bernoulli equation? Give some examples of application.

10. Rotating fluid in a uniform gravitational potential

Consider an ideal fluid contained in a straight cylindrical vessel which rotates with constant angular velocity $\vec{\Omega} = \Omega\,\vec{e}_3$ about its vertical axis, the whole system being placed in a uniform gravitational field $-g\,\vec{e}_3$. Assuming that the fluid rotates with the same angular velocity and that its motion is incompressible, determine the shape of the free surface of the fluid.

Hint: Landau & Lifshitz, *Fluid dynamics* § 10.

11. Differential form of Thomson's theorem

This appellation will become clearer after the lectures in the first week of May. The content of these lectures is not needed to solve the exercise.

i. Consider the motion of an ideal fluid in a gravitation potential. Show that the vorticity vector field $\vec{\omega}(t,\vec{r})$ obeys the equation

$$\frac{\partial \vec{\omega}(t,\vec{r})}{\partial t} = \vec{\nabla} \times \big[\vec{\mathrm{v}}(t,\vec{r}) \times \vec{\omega}(t,\vec{r})\big].$$

Hint: Use the conservation of the entropy per particle to transform the pressure term in the Euler equation.

ii. Stationary vortex: Let $\vec{\omega}(t,\vec{r}) = A\,\delta(x^1)\,\delta(x^2)\,\vec{e}_3$ be the vorticity field in a fluid, with A a real constant and $\{x^i\}$ Cartesian coordinates. Determine the corresponding flow velocity field $\vec{\mathrm{v}}(t,\vec{r})$.

Hint: You should invoke symmetry arguments and Stokes' theorem. A useful formal analogy is provided by the Maxwell–Ampère equation of magnetostatics.

12. Simplified model of star

In an oversimplified approach, one may model a star as a sphere of fluid—a plasma—with uniform mass density ρ. This fluid is in mechanical equilibrium under the influence of pressure $\mathcal{P}$ and gravity. Throughout this exercise, the rotation of the star is neglected.

i. Determine the gravitational field at a distance r from the center of the star.

ii. Assuming that the pressure only depend on r, write down the equation expressing the mechanical equilibrium of the fluid. Determine the resulting function $\mathcal{P}(r)$. Compute the pressure at the star center as function of the mass M and radius R of the star. Calculate the numerical value of this pressure for $M = 2 \times 10^{30}$ kg (solar mass) and $R = 7 \times 10^8$ m (solar radius).

iii. The matter constituting the star is assumed to be an electrically neutral mixture of hydrogen nuclei and electrons. Show that the order of magnitude of the total particle number density of that plasma is $n \approx 2\rho/m_p$, with m_p the proton mass. Estimate the temperature at the center of the sun.

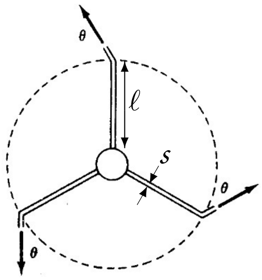
Hint: $m_p = 1.6 \times 10^{-27}$ kg; $k_\mathrm{B} = 1.38 \times 10^{-23}$ J·K^{-1}.

Tutorial sheet 5

Discussion topic: What is Kelvin's circulation theorem? What does it imply for the vorticity?

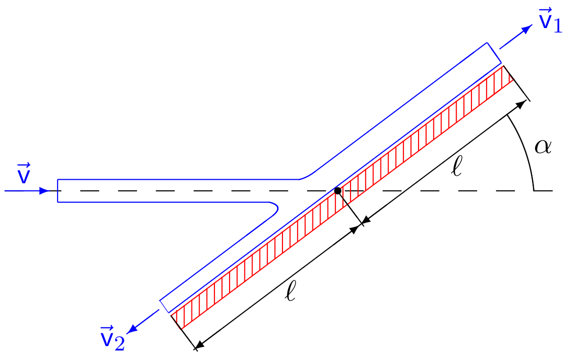
13. Water sprinkler

The horizontal lawn sprinkler schematized below is fed water through its center with a mass flow rate Q. Assuming that water is a perfect incompressible fluid, determine the steady rotation rate as function of Q, the cross section area s of the pipes, their length ℓ, and the angle θ of the emerging water jets with respect to the respective pipes.



14. Water jet

A horizontal jet of water with cross section area $\mathcal{S} = 20\ \mathrm{cm}^2$ and velocity $\mathsf{v} = 20\,\mathrm{m\cdot s^{-1}}$ hits an inclined board of length $2\ell = 20$ cm making an angle α with the horizontal direction, and splits into two jets 1 and 2. The resulting flow is assumed to be steady and incompressible, and water is modeled as a perfect fluid.



i. Show that the influence of gravity on the velocities v_1, v_2 is negligible, so that you can forget it when applying the equation appropriate for the flow under study (which you should apply at the water/air boundary).

ii. Knowing that the force $\vec{F}$ exerted by the water on the board is normal to the latter (why?), determine the cross-section areas $\mathcal{S}_1$, $\mathcal{S}_2$ of the jets as functions of $\mathcal{S}$ and the angle α.

iii. Determine the force $\vec{F}$ and compute the numerical value of $|\vec{F}|$ for $\alpha = 30^{\mathrm{o}}$.

15. Model of a tornado

In a simplified approach, one may model a tornado as the steady incompressible flow of a perfect fluid—air—with mass density $\rho = 1.3\ \mathrm{kg\cdot m^{-3}}$, with a vorticity $\vec{\omega}(\vec{r}) = \omega(\vec{r})\,\vec{e}_3$ which remains uniform inside a cylinder—the "eye" of the tornado—with (vertical) axis along $\vec{e}_3$ and a finite radius $a = 50$ m, and vanishes outside.

i. Express the velocity $\mathsf{v}(r) \equiv \left|\vec{\mathsf{v}}(\vec{r})\right|$ at a distance $r = |\vec{r}|$ from the axis as a function of r and and the velocity $\mathsf{v}_a \equiv \mathsf{v}(r\!=\!a)$ at the edge of the eye.
Compute ω inside the eye, assuming $\mathsf{v}_a = 180$ km/h.

ii. Show that for $r > a$ the tornado is equivalent to a vortex at $x^1 = x^2 = 0$ (as in exercise **11.ii**). What is the circulation around a closed curve circling this equivalent vortex?

iii. Assuming that the pressure $\mathcal{P}$ far from the tornado equals the "normal" atmospheric pressure $\mathcal{P}_0$, determine $\mathcal{P}(r)$ for $r > a$. Compute the barometric depression $\Delta\mathcal{P} \equiv \mathcal{P}_0 - \mathcal{P}$ at the edge of the eye.
Consider a horizontal roof made of a material with mass surface density 100 kg/m^2: is it endangered by the tornado?

16. Vortex sheet

Consider a flow for which the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{\mathsf{v}}$ is large in a thin layer of thickness δ. If the product $\delta \cdot \vec{\omega}$ remains finite—and converges towards a vector $\vec{\varpi}$ in the plane *tangent* to the layer—when $\delta \to 0$, the surface to which the layer shrinks in that limit is referred to as *vortex sheet*.

Prove that if some surface $\mathcal{S}$ is either a vortex sheet, or a surface at which the tangential component of the velocity is discontinuous, then $\vec{\varpi} \times \vec{\mathrm{e}}_n = [\![\vec{\mathsf{v}}_{\|}]\!]$ with $\vec{\mathrm{e}}_n$ the (local) unit normal vector to $\mathcal{S}$ and $[\![\vec{\mathsf{v}}_{\|}]\!]$ the (local) jump of the velocity component tangential to $\mathcal{S}$. Consequently, a vortex sheet is a surface of tangential discontinuity of the velocity, and reciprocally.
Vortex sheets arise for example in the flow around the wing of an airplane

Tutorial sheet 6

Discussion topic: What is a potential flow? What are the corresponding equations of motion?

17. Statics of rotating fluids

This exercise is strongly inspired by Chapter 13.3.3 of the lecture notes on *Applications of Classical Physics* by Roger D. Blandford and Kip S. Thorne.

Consider a fluid, bound by gravity, which is rotating rigidly, i.e. with a uniform angular velocity $\vec{\Omega}_0$ with respect to an inertial frame, around a given axis. In a reference frame that co-rotates with the fluid, the latter is at rest, and thus governed by the laws of hydrostatics—except that you now have to consider an additional term…

i. Relying on your knowledge from point mechanics, show that the usual equation of hydrostatics (in an inertial frame) is replaced in the co-rotating frame by

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\big[\Phi(\vec{r}) + \Phi_{\rm cen.}(\vec{r})\big], \tag{1}$$

where $\Phi_{\rm cen.}(\vec{r}) \equiv -\frac{1}{2}\big[\vec{\Omega}_0 \times \vec{r}\big]^2$ denotes the potential energy from which the centrifugal inertial force (density) derives, $\vec{f}_{\rm cen.} = -\rho\vec{\nabla}\Phi_{\rm cen.}$, while $\Phi(\vec{r})$ is the gravitational potential energy.

ii. Show that Eq. (1) implies that the equipotential lines of $\Phi + \Phi_{\rm cen.}$ coincide with the contours of constant mass density as well as with the isobars.

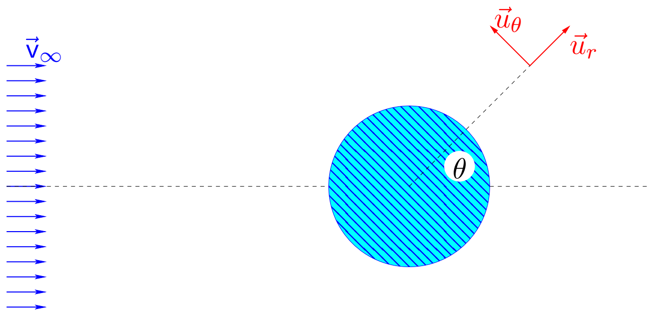
iii. Consider a slowly spinning fluid planet of mass M, assuming for the sake of simplicity that the mass is concentrated at the planet center, so that the gravitational potential is unaffected by the rotation. Let R_e resp. R_p denote the equatorial resp. polar radius of the planet, where $|R_e - R_p| \ll R_e \simeq R_p$, and g be the gravitational acceleration at the surface of the planet.
Using questions **i.** and **ii.**, show that the difference between the equatorial and polar radii is

$$R_e - R_p \simeq \frac{R_e^2|\vec{\Omega}_0|^2}{2g}.$$

Compute this difference in the case of Earth ($R_e \simeq 6.4\times10^3$ km)—which as everyone knows behaves as a fluid if you look at it long enough—and compare with the actual value.

18. Potential flow with a vortex. Magnus effect

The purpose of this exercise is to introduce a simplified model for the Magnus effect, which was discussed in the lectures.



One can show (cf. Landau–Lifshitz, *Fluid dynamics*, § 10) that the flow velocity of an incompressible perfect fluid around a cylinder of radius R at rest, with the uniform condition $\vec{\mathsf{v}}(\vec{r}) = \vec{\mathsf{v}}_\infty$ far from the cylinder—$\vec{\mathsf{v}}_\infty$ being perpendicular to the cylinder axis—, is given by

$$\vec{\mathsf{v}}(r,\theta) = \mathsf{v}_\infty\left[\left(1-\frac{R^2}{r^2}\right)\cos\theta\,\vec{u}_r - \left(1+\frac{R^2}{r^2}\right)\sin\theta\,\vec{u}_\theta\right], \tag{2}$$

where (r,θ) are polar coordinates—the third dimension (z), along the cylinder axis, plays no role—with the origin at the center of the cylinder (see Figure) and $\vec{u}_r$, $\vec{u}_\theta$ unit length vectors.

One superposes to the velocity field (2) a vortex with circulation Γ, corresponding to a flow velocity

$$\vec{\mathsf{v}}(r,\theta) = \frac{\Gamma}{2\pi r}\vec{u}_\theta. \tag{3}$$

i. Let $C \equiv \Gamma/(4\pi R\mathsf{v}_\infty)$. Determine the points with vanishing velocity for the flow resulting from superposing (2) and (3).

Hint: Distinguish the two cases $C < 1$ and $C > 1$.

ii. How do the streamlines look like in each case? Comment on the physical meaning of the result.

iii. Express the force per unit length $\mathrm{d}\vec{F}/\mathrm{d}z$ exerted on the cylinder by the flow (2)+(3) as function of Γ, v_∞ and the mass density ρ of the fluid.

19. Two-dimensional potential flow. Teapot effect

dedicated mit freundlichen Grüßen to T.L., for whom this exercise was the highlight of the lectures.

Consider a two-dimensional potential flow with velocity $\vec{\mathsf{v}}(t,x,y)$, with (x,y) Cartesian coordinates. Let $\varphi(t,x,y)$ be the corresponding velocity potential ($\vec{\mathsf{v}} = -\vec{\nabla}\varphi$) and $\psi(t,x,y)$ the so-called "stream function", such that $\mathsf{v}_x = -\partial_y\psi$ and $\mathsf{v}_y = \partial_x\psi$. Define a complex variable z by $z = x + \mathrm{i}y$.

i. Show that the complex potential defined by $\phi = \varphi + \mathrm{i}\psi$ is a holomorphic/analytic function of z, by checking that the Cauchy–Riemann equations hold.

ii. Show that the stream function obeys the Laplace differential equation and that the lines of constant $\psi(t,x,y)$ are the streamlines of the flow.

iii. Check that the "complex velocity" $\mathsf{w} \equiv -\dfrac{\mathrm{d}\phi}{\mathrm{d}z}$ equals $\mathsf{v}_x - \mathrm{i}\mathsf{v}_y$.

iv. Consider the complex potential $\phi(z) = Az^n$ with $A \in \mathbb{R}$ and $n \geq 1/2$. Show that this potential allows you to describe the flow in the sector $\widehat{\mathcal{E}}$ delimited by two walls making an angle $\alpha = \pi/n$.

Hint: Landau–Lifshitz, *Fluid dynamics*, § 10.

v. What can you say about the flow velocity in the vicinity of the end-corner of the sector $\widehat{\mathcal{E}}$?

Hint: Distinguish the cases $\alpha < \pi$ and $\alpha > \pi$.

vi. Teapot effect

If one tries to pour tea "carefully" from a teapot, one will observe that the liquid will trickle along the lower side of the nozzle, instead of falling down into the cup waiting below. Explain this phenomenon using the flow profile introduced above (in the case $\alpha > \pi$) and the Bernoulli equation.

Literature: Jearl Walker, Scientific American, Oct. 1984 (= Spektrum der Wissenschaft, Feb. 1985).

vii. Assuming now that you are using the potential $\phi(z) = Az^n$ to model the flow of a river, which qualitative behavior can you anticipate for its bank?

Tutorial sheet 7

Discussion topics: What is a sound wave? How do you derive the corresponding equation of motion? How is the speed of sound defined? What happens when the amplitude of the wave becomes large?

20. One-dimensional "similarity flow"

Consider a perfect fluid at rest in the region $x \geq 0$ with pressure $\mathcal{P}_0$ and mass density ρ_0; the region $x < 0$ is empty ($\mathcal{P} = 0$, $\rho = 0$). At time $t = 0$, the wall separating both regions is removed, so that the fluid starts flowing into the region $x < 0$. The goal of this exercise is to solve this instance of *Riemann's problem* by determining the flow velocity $\mathsf{v}(t, x)$ for $t > 0$. It will be assumed that the pressure and mass density of the fluid remain related by

$$\frac{\mathcal{P}}{\mathcal{P}_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \qquad \text{with } \gamma > 1$$

throughout the motion. This relation also gives you the speed of sound $c_s(\rho)$.

i. Assume that the dependence on t and x of the various fields involves only the combination $u \equiv x/t$.[1] Show that the continuity and Euler equations can be recast as

$$\big[u - \mathsf{v}(u)\big]\,\rho'(u) = \rho(u)\,\mathsf{v}'(u)$$
$$\rho(u)\big[u - \mathsf{v}(u)\big]\,\mathsf{v}'(u) = c_s^2(\rho(u))\,\rho'(u),$$

where ρ' resp. v' denote the derivative of ρ resp. v with respect to u.

ii. Show that the velocity is either constant, or obeys the equation $u - \mathsf{v}(u) = c_s(\rho(u))$, in which case the squared speed of sound takes the form $c_s^2(\rho) = c_s^2(\rho_0)(\rho/\rho_0)^{\gamma-1}$.

iii. Show that the results of **i.** and **ii.** lead to the relation

$$\mathsf{v}(u) = a + \frac{2}{\gamma - 1}\,c_s(\rho(u)),$$

where a denotes a constant whose value is fixed by the condition that $\mathsf{v}(u)$ remain continuous inside the fluid. Show eventually that in some interval for the values of u, the norm of v is given by

$$|\mathsf{v}(u)| = \frac{2}{\gamma + 1}\big[c_s(\rho_0) - u\big].$$

iv. Sketch the profiles of the mass density $\rho(u)$ and the streamlines $x(t)$ and show that after the removal of the separation at $x = 0$ the information propagates with velocity $2c_s(\rho_0)/(\gamma - 1)$ towards the negative-x region, while it moves to the right with the speed of sound $c_s(\rho)$.

21. Inviscid Burgers equation

The purpose of this exercise is to show how an innocent-looking—yet non-linear—partial differential equation with a smooth initial condition may lead after finite amount of time to a discontinuity, i.e. a shock wave.

Neglecting the pressure term in the one-dimensional Euler equation leads to the so-called *inviscid Burgers equation*

$$\frac{\partial \mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\,\frac{\partial \mathsf{v}(t,x)}{\partial x} = 0.$$

i. Show that the solution with (arbitrary) given initial condition $\mathsf{v}(0, x)$ for $x \in \mathbb{R}$ obeys the implicit equation $\mathsf{v}(0, x) = \mathsf{v}\big(t, x + \mathsf{v}(0, x)\,t\big)$.

Hint: `http://en.wikipedia.org/wiki/Burgers'_equation`

[1]... which is what is meant by "self-similar".

ii. Consider the initial condition $\mathsf{v}(0,x) = \mathsf{v}_0\,\mathrm{e}^{-(x/x_0)^2}$ with v_0 and x_0 two real numbers. Show that the flow velocity becomes discontinuous at time $t = \sqrt{\mathrm{e}/2}\,x_0/\mathsf{v}_0$, namely at $x = x_0\sqrt{2}$.

22. (1+1)-dimensional relativistic motion

On May 28th, the flow velocities considered in the lectures will reach the relativistic regime. To prepare for this event, you may refresh your knowledge on Special Relativity. This exercise is here to help you in that direction, and also introduces coordinates which will be used later in the lectures.

Consider a (1+1)-dimensional relativistic motion along a direction denoted as x, where "1+1" stands for one time and one spatial dimension. A first possibility is to use Minkowski $(x^0, x^1) = (t, x)$ coordinates, such that the metric tensor has components $g_{00} \equiv g_{tt} = -1$, $g_{11} \equiv g_{xx} = +1$, $g_{01} = g_{10} = 0$.[2] If there is a high-velocity motion in the x-direction, a better choice might be to used the *proper time* (of a comoving observer) τ and *spatial rapidity* ς such that

$$x^{0'} \equiv \tau \equiv \sqrt{t^2 - x^2}, \quad x^{1'} \equiv \varsigma \equiv \frac{1}{2}\log\frac{t+x}{t-x} \quad \text{where } |x| \leq t.$$

Throughout, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein's summation convention.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau\cosh\varsigma, \quad x = \tau\sinh\varsigma.$$

(*Hint:* Recognize $\frac{1}{2}\log\frac{1+u}{1-u}$).
Deduce the following relationship between the basis vectors of the two coordinate systems

$$\begin{cases} \vec{\mathrm{e}}_\tau(\tau,\varsigma) = \cosh\varsigma\,\vec{\mathrm{e}}_t + \sinh\varsigma\,\vec{\mathrm{e}}_x \\ \vec{\mathrm{e}}_\varsigma(\tau,\varsigma) = \tau\sinh\varsigma\,\vec{\mathrm{e}}_t + \tau\cosh\varsigma\,\vec{\mathrm{e}}_x \end{cases}$$

and write down the metric tensor $g_{0'0'} \equiv g_{\tau\tau}$, $g_{1'1'} \equiv g_{\varsigma\varsigma}\dots$ in the new coordinate system.
For the sake of completeness, give also the components $g^{\mu'\nu'}$ of the inverse metric tensor.

ii. Inspiring yourself from what was done in the case of the two-dimensional Euclidean plane in the lecture, compute the Christoffel symbols $\Gamma^{\mu'}_{\nu'\rho'}$ where the primed indices run over all values $0'$, $1'$.

iii. Let N^μ denote the components of a "2-vector".
Write down the covariant derivative $\mathrm{d}_{\nu'}N^{\mu'} \equiv \mathrm{d}N^{\mu'}/\mathrm{d}x^{\nu'}$ that generalizes to curvilinear (τ,ς) coordinates the derivative $\partial_\nu N^\mu \equiv \partial N^\mu/\partial x^\nu$ of Minkowski coordinates. Compute the "2-divergence" $\mathrm{d}_{\mu'}N^{\mu'}$.

iv. Let $T^{\mu\nu}$ denote the components of a symmetric $\binom{2}{0}$-tensor, such that $T^{01} = 0$.
Write down the covariant derivative $\mathrm{d}_{\rho'}T^{\mu'\nu'}$ and compute $\mathrm{d}_{\mu'}T^{\mu'\nu'}$ for $\nu' \in \{\tau,\varsigma\}$.

v. Draw on a spacetime diagram—with t on the vertical axis and x on the horizontal axis—the lines of constant τ and those of constant ς.

Remark: The coordinates (τ,σ) are sometimes called *Milne coordinates*.

[2] Note that I shall use the "mostly plus" convention for the metric tensor, in which timelike vectors have a negative semi-norm.

Tutorial sheet 8

Discussion topics: What are the fundamental equations of the dynamics of a relativistic fluid? What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?

23. Propagation of internal waves in the ocean

This exercise is inspired by an article (in French) by L. Gostiaux and T. Dauxois published in the *Bulletin de l'Union des Professeurs de Physique et de Chimie* (Nov. 2004)

The properties of several important instances of fluids found in nature—in particular their mass density ρ—depend on the altitude/depth z (oriented upwards): these fluids are said to be *stratified*. In the example of ocean water, ρ depends on depth "directly", i.e. because it is a function of pressure $\mathcal{P}$ which depends itself on z, but also "indirectly", inasmusch as depth influences the salinity[1] [concentration in salt(s)], which in turn affects ρ.

The purpose of this exercise is to investigate internal waves in a stratified fluid at rest and in particular to exemplify a rather unusual dispersive behavior. Throughout, we consider a two-dimensional problem; as in the lecture, "equilibrium" quantities, related to the unperturbed fluid in absence of wave, are denoted with a subscript 0.

i. Brunt–Väisälä frequency

If a fluid particle is displaced vertically from its equilibrium position z_0 by an amount δz quickly enough, it will evolve adiabatically and without adjusting its salinity, so that when it is at $z_0 + \delta z$, its mass density ρ' differs from the equilibrium mass density $\rho_0(z_0 + \delta z)$ at that depth.

a) Considering the forces acting on the displaced fluid particle, show that Newton's second law gives

$$\rho' \frac{\mathrm{d}^2\delta z}{\mathrm{d}t^2} = -g\big[\rho' - \rho_0(z_0 + \delta z)\big], \tag{1}$$

with g the acceleration due to gravity.

The *Boussinesq approximation*, which will also be used in question **ii.**, consists in approximating the mass density in the inertial term [left hand side of Eq. (1)] by the equilibrium value $\rho_0(z_0)$, while still keeping the "exact" value (here ρ') in the force term.

b) For the right-hand side of Eq. (1), one introduces a "potential density"[2] $\bar\rho$—which equals the mass density ρ under the same conditions of temperature and salinity, yet at a fixed reference pressure—such that the difference in the term between square brackets can be recast as $-(\mathrm{d}\bar\rho/\mathrm{d}z)\delta z$.

Under which condition on the derivative $\mathrm{d}\bar\rho/\mathrm{d}z$ is the equilibrium of the stratified fluid stable? In that case, what is the motion of the fluid particle? You may find it interesting to introduce the *Brunt–Väisälä "frequency"* defined by the relation $\omega^2_{\text{B–V}} \equiv -(g/\rho_0)\,\mathrm{d}\bar\rho/\mathrm{d}z$.

ii. Propagation of internal waves

Starting from a state of (stratified) rest, we consider small perturbations $\delta\rho(t,x,z)$, $\delta\mathcal{P}(t,x,z)$, $\delta\vec{\mathsf{v}}(t,x,z)$, assuming that the resulting flow is incompressible. We shall assume that the conservation of the potential density along streamlines reads

$$\frac{\partial\delta\rho}{\partial t} + \delta\mathsf{v}_z \frac{\mathrm{d}\bar\rho}{\mathrm{d}z} = 0. \tag{2}$$

[1] Bonus question to the former "Non-equilibrium physics" students: how? (neglecting the temperature variations)

[2] For more information, see `https://en.wikipedia.org/wiki/Potential_density` or `http://oceanworld.tamu.edu/resources/ocng_textbook/chapter06/chapter06_05.htm`.

a) Write down the (kinematic) incompressibility condition, which will be hereafter referred to as Eq. (3).

b) Show that the Euler equation in the Boussinesq approximation introduced above gives you to leading order the usual fundamental equation of hydrostatics, while the subleading order yields

$$\rho_0 \frac{\partial \delta \mathsf{v}_x}{\partial t} = -\frac{\partial \delta \mathcal{P}}{\partial x} \quad , \quad \rho_0 \frac{\partial \delta \mathsf{v}_z}{\partial t} = -\frac{\partial \delta \mathcal{P}}{\partial z} - \delta\rho\, g. \tag{4,5}$$

Together with Eqs. (2) and (3), you now have four equations for the four unknown fields $\delta\rho$, $\delta\mathcal{P}$, $\delta\mathsf{v}_x$, and $\delta\mathsf{v}_z$.

c) Neglecting the spatial variations of ρ_0—which seems at first paradoxical, but amounts to assuming that the typical length scale of variations $|g/\omega^2_{\rm B-V}|$ is much larger than the vertical wavelength of waves—show that equations (2)–(5) lead to the differential equation

$$\frac{\partial^4 \delta \mathsf{v}_z}{\partial t^2\, \partial x^2} + \frac{\partial^4 \delta \mathsf{v}_z}{\partial t^2\, \partial z^2} = -\omega^2_{\rm B-V} \frac{\partial^2 \delta \mathsf{v}_z}{\partial x^2}. \tag{6}$$

[*Hint*: consider (2) and (5) on the one hand, (3) and (4) on the other hand]

Convince yourself that $\delta\mathsf{v}_x$, $\delta\mathcal{P}$ and $\delta\rho$ obey similar equations.

d) Show that the harmonic ansatz $\delta\mathsf{v}_z = \widetilde{\delta\mathsf{v}_z}\,{\rm e}^{-{\rm i}(\omega t - \vec{k}\cdot\vec{r})}$ leads to the dispersion relation

$$\omega = \pm\omega_{\rm B-V} \sin\beta_{\vec{k}}, \tag{7}$$

where $\beta_{\vec{k}}$ is the angle between the wave vector $\vec{k}$ and the z-direction. What do you find surprising here? Discuss the physics for various values of ω (consider 4 cases!).

e) Compute the phase velocity $\vec{c}_\varphi(\vec{k})$ (remember that it is directed along $\vec{k}$) and the group velocity $\vec{c}_g(\vec{k}) \equiv {\rm d}\omega/{\rm d}\vec{k}$ following from the dispersion relation (7), and compare them with each other. Your result begs for comments!

24. Energy-momentum tensor

Let $\mathcal{R}$ denote a fixed reference frame. Show with the help of a Lorentz transformation that the Minkowski coordinates of the (local) energy-momentum tensor of a perfect fluid whose local rest frame moves with velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}$, are given to order $\mathcal{O}(|\vec{\mathsf{v}}|/c)$ by

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & \mathcal{P} & 0 & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & 0 & \mathcal{P} & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} & 0 & 0 & \mathcal{P} \end{pmatrix}.$$

Check the compatibility of this result with the general formula for $T^{\mu\nu}$ given in the lecture.

25. Isentropic flow

Let s resp. n denote the entropy density resp. particle number density, and u^μ the flow velocity. The entropy per particle is defined as $s/n = S/N$. Show that in an isentropic flow $[{\rm d}_\mu(su^\mu) = 0]$ the entropy per particle is conserved, i.e. ${\rm d}(s/n)/{\rm d}t = 0$.

Hint: If the covariant derivatives ${\rm d}_\mu$ upset you, choose Minkowski coordinates, in which ${\rm d}_\mu = \partial_\mu$.

Tutorial sheet 9

26. Equations of motion of a perfect relativistic fluid

In this exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

i. Show that the tensor with components $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ defines a projector on the subspace orthogonal to the 4-velocity.
Denoting by d_μ the components of the (covariant) 4-gradient, we define $\nabla^\nu \equiv \Delta^{\mu\nu}\mathrm{d}_\mu$. Can you see the rationale behind this notation?

ii. Show that the energy-momentum conservation equation for a perfect fluid is equivalent to the two equations

$$u^\mu \mathrm{d}_\mu \epsilon + (\epsilon + \mathcal{P})\mathrm{d}_\mu u^\mu = 0 \quad \text{and} \quad (\epsilon + \mathcal{P})u^\mu \mathrm{d}_\mu u^\nu + \nabla^\nu \mathcal{P} = 0.$$

Which known equation does the second one evoke?

27. Particle number conservation

Consider a 4-current with components $N^\mu(x)$ obeying the continuity equation $\mathrm{d}_\mu N^\mu(\mathsf{x}) = 0$. Show that the quantity $\mathcal{N} = \int N^0(\mathsf{x})\,\mathrm{d}^3\vec{r}/c$ is a Lorentz scalar, by convincing yourself first that $\mathcal{N}$ can be rewritten in the form

$$\mathcal{N} = \frac{1}{c}\int_{x^0=\text{const.}} N^\mu(\mathsf{x})\,\mathrm{d}^3\sigma_\mu, \tag{1}$$

where $\mathrm{d}^3\sigma_\mu = \dfrac{1}{6}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ is a 4-vector, with $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ the antisymmetric 4-tensor defined by

$$\mathrm{d}^3\mathcal{V}^{012} = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2, \quad \mathrm{d}^3\mathcal{V}^{021} = -\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^1, \quad \text{etc.}$$

and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric Levi–Civita tensor with the convention $\epsilon_{0123} = +1$, such that $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ represents the 3-dimensional hypersurface element in Minkowski space.

28. Speed of sound in ultrarelativistic matter

Consider a perfect fluid with the usual energy-momentum tensor. $T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (\epsilon + \mathcal{P})u^\mu u^\nu/c^2$. It is assumed that there is no conserved particle number relevant for thermodynamics, so that the energy density in the local rest frame ϵ is function of a single thermodynamic variable, for instance $\epsilon = \epsilon(\mathcal{P})$. Throughout the exercise, Minkowski coordinates are used.

A background "flow" with local-rest-frame energy density and pressure ϵ_0 and $\mathcal{P}_0$ is submitted to a small perturbation resulting in $\epsilon = \epsilon_0 + \delta\epsilon$, $\mathcal{P} = \mathcal{P}_0 + \delta\mathcal{P}$, and $\vec{\mathsf{v}} = \vec{0} + \delta\vec{\mathsf{v}}$.

i. Starting from the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, show that linearization to first order in the perturbations leads to the two equations of motion $\partial_t\delta\epsilon = -(\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\delta\vec{\mathsf{v}}$ and $(\epsilon_0 + \mathcal{P}_0)\partial_t\delta\vec{\mathsf{v}} = -c^2\vec{\nabla}\delta\mathcal{P}$.

ii. Show that the speed of sound is given by the expression $c_s^2 = \dfrac{c^2}{\mathrm{d}\epsilon/\mathrm{d}\mathcal{P}}$.

iii. Compute c_s for a fluid obeying the Stefan–Boltzmann law[1] $\mathcal{P} = \dfrac{g\pi^2}{90}\dfrac{(k_\mathrm{B}T)^4}{(\hbar c)^3}$, with g the number of degrees of freedom (e.g. $g = 2$ for blackbody radiation).

Hint: You may find the Gibbs–Duhem relation useful...

[1] This is a good opportunity to refresh your knowledge on the statistical physics of relativistic systems. Can you give a physical argument why quantum effects always play a role in such systems, as signaled by the presence of $\hbar$ in the equation of state?

Tutorial sheet 10

Discussion topic: What are the fundamental equations governing the dynamics of non-relativistic Newtonian fluids?

29. One-dimensional relativistic flow

In the lecture on the 9th of June, the equations describing the "boost-invariant" one-dimensional expansion of a perfect relativistic fluid were presented. Here, we investigate another one-dimensional solution of the equations of relativistic fluid dynamics, originally due to L. Landau [Izv. Akad. Nauk. USSR **17** (1953) 51], again for a medium without conserved quantum number.

Throughout the exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

i. Considering a one-dimensional expansion along the z-axis, write down the non-trivial equations of motion expressing energy-momentum conservation in Minkowski coordinates.
From now on, the equation of state of the expanding perfect fluid is assumed to be $\epsilon = 3\mathcal{P}$.

ii. The so-called *light-cone coordinates* are defined as $x^+ \equiv \dfrac{t+z}{\sqrt{2}}$, $x^- \equiv \dfrac{t-z}{\sqrt{2}}$, where the factor $1/\sqrt{2}$ is not universal, yet convenient.

a) Although this is irrelevant for the rest of the exercise, write down the metric tensor in light-cone coordinates. Of which type are the basis vectors in the x^+ and x^- directions?

b) Check the identities $\dfrac{\partial}{\partial t} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x^+} + \dfrac{\partial}{\partial x^-}\right)$ and $\dfrac{\partial}{\partial z} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x^+} - \dfrac{\partial}{\partial x^-}\right)$.

c) Show that they allow you to transform the equations of motion of question **i.** into

$$\frac{\partial\epsilon}{\partial x^+} + 2\frac{\partial(\epsilon\mathrm{e}^{-2y_\mathrm{f}})}{\partial x^-} = 0 \quad , \quad 2\frac{\partial(\epsilon\mathrm{e}^{2y_\mathrm{f}})}{\partial x^+} + \frac{\partial\epsilon}{\partial x^-} = 0, \tag{1}$$

with $y_\mathrm{f}(\mathsf{x})$ the position-dependent "flow rapidity", such that $u^t = \cosh y_\mathrm{f}$, $u^z = \sinh y_\mathrm{f}$.

iii. Let $y_\pm \equiv \log\dfrac{x^\pm}{\Delta}$, with Δ some length scale.

a) Show that the expansion with energy density and flow rapidity

$$\epsilon(y_+, y_-) = \epsilon_0 \exp\left[-\frac{4}{3}\left(y_+ + y_- - \sqrt{y_+ y_-}\right)\right] \quad , \quad y_\mathrm{f}(y_+, y_-) = \frac{y_+ - y_-}{2} \tag{2}$$

is solution to the equations of motion (1), with ϵ_0 a constant.
Hint: The identity $\mathrm{e}^{2y_\mathrm{f}(y_+,y_-)} = x^+/x^-$ may be helpful.

b) Transforming the previous solution (2) back to Minkowski variables, sketch the energy density (in units of ϵ_0) and the flow rapidity as a function of z at successive instants $t = \Delta, 2\Delta, 4\Delta, 8\Delta$ for $|z| < t$. Can you guess what physical problem Landau was trying to model?

30. Heat diffusion

In the lecture (June 11), we derived the equation

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla}\cdot\left[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\right]$$

valid in a dissipative fluid at rest, with κ the heat capacity.

Assuming that $C \equiv \partial e/\partial T$ and κ are constant coefficients and introducing $\chi \equiv \kappa/C$, determine the temperature profile $T(t,\vec{r})$ for $z < 0$ with the boundary condition of a uniform temperature in the plane $z = 0$, which evolves in time as $T(t, z=0) = T_0 \cos(\omega t)$. At which depth is the amplitude of the temperature oscillations 10 % of that in the plane $z = 0$?

31. Taylor–Couette flow. Measurement of shear viscosity

A Couette viscometer consists of an annular gap, filled with fluid, between two concentric cylinders with height L. The outer cylinder (radius R_2) rotates around the common axis with angular velocity Ω_2, while the inner cylinder (radius R_1) remains motionless. The motion of the fluid is assumed to be two-dimensional, incompressible and steady.

i. Check that the continuity equation leads to $\mathsf{v}^r = 0$, with v^r the radial component (in a system of cylinder coordinates) of the flow velocity.

ii. Prove that the Navier–Stokes equation lead to the equations

$$\frac{\mathsf{v}^\varphi(r)^2}{r} = \frac{1}{\rho}\frac{\partial \mathcal{P}(r)}{\partial r} \tag{3}$$

$$\frac{\partial^2 \mathsf{v}^\varphi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \mathsf{v}^\varphi(r)}{\partial r} - \frac{\mathsf{v}^\varphi(r)}{r^2} = 0. \tag{4}$$

What is the meaning of Eq. (3)? Solve Eq. (4) with the ansatz $\mathsf{v}^\varphi(r) = ar + \frac{b}{r}$.

iii. One can show (can you?) that the $r\varphi$-component of the stress tensor is given by

$$\sigma^{r\varphi} = \eta\left(\frac{1}{r}\frac{\partial \mathsf{v}^r}{\partial \varphi} + \frac{\partial \mathsf{v}^\varphi}{\partial r} - \frac{\mathsf{v}^\varphi}{r}\right).$$

Show that $\sigma^{r\varphi} = -\frac{2b\eta}{r^2}$, where b is the same coefficient as above.

iv. A torque $\mathcal{M}_z$ is measured at the surface of the inner cylinder. How can the shear viscosity η of the fluid be deduced from this measurement?
Numerical example: $R_1 = 10$ cm, $R_2 = 11$ cm, $L = 10$ cm, $\Omega_2 = 10$ rad·s^{-1} and $\mathcal{M}_z = 7,246\cdot 10^{-3}$ N·m.

Tutorial sheet 11

Discussion topic: Dynamical similarity and the Reynolds number

32. Flow due to an oscillating plane boundary

Consider a rigid infinitely extended plane boundary ($y = 0$) that oscillates in its own plane with a sinusoidal velocity $U\cos(\omega t)\,\vec{e}_x$. The region $y > 0$ is filled with an incompressible Newtonian fluid with uniform kinematic shear viscosity ν. We shall assume that volume forces on the fluid are negligible, that the pressure is uniform and remains constant in time, and that the fluid motion induced by the plane oscillations do not depend on the coordinates x, z.

i. Determine the flow velocity $\vec{\mathsf{v}}(t, y)$ and plot the resulting profile.

ii. What is the characteristic thickness of the fluid layer in the vicinity of the plane boundary that follows the oscillations? Comment on your result.

33. Dimensional consideration for viscous flows in a tube

Consider the motion of a given fluid in a cylindrical tube of length L and of circular cross section under the action of a difference $\Delta\mathcal{P}$ between the pressures at the two ends of the tube. The relation between the pressure drop per unit length $\Delta\mathcal{P}/L$ and the magnitude of the mean velocity $\langle\mathsf{v}\rangle$—defined as the average over a cross section of the tube—is given by

$$\frac{\Delta\mathcal{P}}{L} = C\langle\mathsf{v}\rangle^n,$$

with C a constant that depends on the fluid mass density ρ, on the kinematic shear viscosity ν, and on the radius a of the tube cross section. n is a number which depends on the type of flow: $n = 1$ if the flow is laminar (this is the Hagen–Poiseuille law seen in the lecture), while measurements in turbulent flows by Hagen (1854) resp. Reynolds (1883) have given $n = 1.75$ resp. $n = 1.722$.

Assuming that C is—up to a pure number—a product of powers of ρ, ν and a, determine the exponents of these power laws using dimensional arguments.

34. Laminar flow of a river

A river is modeled as an incompressible Newtonian fluid flowing steadily and laminarly down a channel—the river bed—with uniform rectangular cross-section inclined at a constant angle α from the horizontal. The river itself is assumed to be a layer of constant thickness h, so that its free surface is a plane parallel to its bottom.

To fix notations, let x denote the direction along which water flows, with the basis vector oriented downstream, and y be the direction perpendicular to the river bed, oriented upwards. The river bottom resp. free surface is at $y = 0$ resp. $y = h$, the vertical edges of the river bed are at $z = \pm b$ with $2b$ the river width.

i. Equations of motion

Assuming that the pressure at the free surface of the water as well as "at the ends" at large $|x|$ is constant—i.e., the river flow is caused by gravity, not by a pressure gradient—, show that the flow velocity magnitude v, pressure $\mathcal{P}$ and mass density ρ of the fluid obey the equations

$$\begin{cases} \dfrac{\partial\mathsf{v}}{\partial x} = 0 \\ \eta\triangle\mathsf{v} = -\rho g\sin\alpha \\ \dfrac{\partial\mathcal{P}}{\partial y} = -\rho g\cos\alpha, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} \mathsf{v} = 0 & \text{at } y = 0 \text{ and at } z = \pm b \\ \dfrac{\partial \mathsf{v}}{\partial y} = 0 & \text{at } y = h \\ \mathcal{P} = \mathcal{P}_0 & \text{at } y = h. \end{cases} \tag{2}$$

You can immediately write down the solution for the pressure.

ii. Flow velocity

For the sake of brevity, the constant $-(\rho g/\eta)\sin\alpha$ will from now on be denoted by c.

a) Check that the ansatz

$$\mathsf{v}(\vec{r}) = \sum_{n=0}^{\infty} f_n(y)\cos(k_n z) \text{ with } k_n \equiv (2n+1)\frac{\pi}{2b} \tag{3}$$

automatically fulfills some of the equations and boundary conditions.

b) The physics we are interested is restricted to the region $-b \leq z \leq b$, so that we can extend any function arbitrarily beyond the river bed edges. Accordingly, a neat trick is to continue the constant c to a periodic—as suggested by ansatz (3)—non-constant function, namely the sum

$$\tilde{c}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{2\,c}{k_n b}\cos(k_n z). \tag{4}$$

Check (plot!) that this function does coincide with c on the interval $-b < z < b$. Do you know how it looks like outside the interval?

c) Write down the (inhomogeneous) differential equation obeyed by each of the functions $f_n(y)$. Solve it under consideration of the as yet unused boundary conditions (2) at $y = 0$ and $y = h$.

Normally, you should find $\mathsf{v}(\vec{r}) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2\,c}{k_n^3 b}\left[\frac{\cosh\big(k_n(y-h)\big)}{\cosh(k_n h)} - 1\right]\cos(k_n z)$.

d) Show the mass flow rate across the river cross section is

$$Q = 4c\left(\sum_{n=0}^{\infty} \frac{\tanh(k_n h)}{k_n^5 b} - \frac{h}{b}\sum_{n=0}^{\infty}\frac{1}{k_n^4}\right).$$

Assuming that the river is much wider than deep, compute the leading order ($\propto bh^3$) contribution to the mass flow rate.

Remark: This leading order coincides with the result in the case where the river bed is unbounded in the z-direction, see Landau & Lifshitz, *Fluid Mechanics*, § 17 Problem 5. The next-to-leading order contribution is unfortunately much harder to obtain, since the Taylor expansion of the tanh term starts giving divergent series. In a 1910 article by a student of Arnold Sommerfeld—who also thanks Peter Debye in his acknowledgments—one may find an expression for this next-to-leading order term, which however seems very dubious, as it is proportional to h^4, while the expression of Q is odd in h. . .

Tutorial sheet 12

Discussion topic: Life at low Reynolds number (educate yourself by reading E. M. Purcell's article) and the "scallop theorem".

35. Equations of fluid dynamics in a uniformly rotating reference frame

This exercise is inspired by Chapter 14.5.1 of the lecture notes on *Applications of Classical Physics* by Roger D. Blandford and Kip S. Thorne.

For the study of various physical problems (see examples in question **iv.a**), it may be more convenient to study the dynamics of a fluid from a reference frame $\mathcal{R}_{\Omega_0}$ in uniform rotation with angular velocity $\vec{\Omega}_0$ with respect to an inertial frame $\mathcal{R}_0$.
In exercise **17**, you already investigated hydrostatics in a rotating reference frame: in that case only the centrifugal acceleration plays a role, which can be entirely recast as the effect of a potential energy $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big(\vec{\Omega}_0 \times \vec{r}\big)^2$ leading to the centrifugal inertial force density $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$. The purpose of this exercise is to generalize that result to the derivation of (some of) the equations governing a flowing Newtonian fluid.

i. Kinematics

Recall the expressions of the centrifugal and Coriolis accelerations acting on a small fluid element in terms of its position vector $\vec{r}$ and velocity $\vec{\mathsf{v}}$ (measured in $\mathcal{R}_{\Omega_0}$) and of the angular velocity.

ii. Incompressibility condition

Writing down the relation between the velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}_{\Omega_0}$ and that measured in $\mathcal{R}_0$, show that the incompressibility condition valid in the inertial frame leads to $\vec{\nabla}\cdot\vec{\mathsf{v}} = 0$.

iii. Navier–Stokes equation

Show that the incompressible Navier–Stokes equation from the point of view of an observer at rest in the rotating reference frame $\mathcal{R}_{\Omega_0}$ reads (the variables are omitted)

$$\frac{\mathrm{D}\vec{\mathsf{v}}}{\mathrm{D}t} = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}_{\text{eff.}} + \nu\triangle\vec{\mathsf{v}} - 2\vec{\Omega}_0 \times \vec{\mathsf{v}} \tag{1}$$

where $\mathcal{P}_{\text{eff.}} = \mathcal{P} + \rho\big(\Phi + \Phi_{\text{cen.}}\big)$, with Φ the potential energy from which (non-inertial) volume forces acting on the fluid derive. Check that you recover the equation of hydrostatics found in exercise **17**.

iv. Dimensionless numbers and limiting cases

a) Let L_c resp. v_c denote a characteristic length resp. velocity for a given flow. The Ekman and Rossby numbers are respectively defined as

$$\text{Ek} \equiv \frac{\nu}{|\Omega_0|L_c^2} \quad , \quad \text{Ro} \equiv \frac{\mathsf{v}_c}{|\Omega_0|L_c}.$$

Compute Ek and Ro in a few numerical examples:
– $L_c \approx 100$ km, $\mathsf{v}_c \approx 10\ \text{m}\cdot\text{s}^{-1}$, $\Omega_0 \approx 10^{-4}\ \text{rad}\cdot\text{s}^{-1}$, $\nu \approx 10^{-5}\ \text{m}^2\cdot\text{s}^{-1}$ (wind in the Earth atmosphere);
– $L_c \approx 1000$ km, $\mathsf{v}_c \approx 0.1\ \text{m}\cdot\text{s}^{-1}$, $\Omega_0 \approx 10^{-4}\ \text{rad}\cdot\text{s}^{-1}$, $\nu \approx 10^{-6}\ \text{m}^2\cdot\text{s}^{-1}$ (ocean stream);
– $L_c \approx 10$ cm, $\mathsf{v}_c \approx 1\ \text{m}\cdot\text{s}^{-1}$, $\Omega_0 \approx 10\ \text{rad}\cdot\text{s}^{-1}$, $\nu \approx 10^{-6}\ \text{m}^2\cdot\text{s}^{-1}$ (coffee/tea in your cup).

b) Assuming stationarity, which term in Eq. (1) is negligible (against which) at small Ekman number? at small Rossby number?

Write down the simplified equation of motion valid when both Ek $\ll 1$ and Ro $\ll 1$ (to which of the above examples does this correspond?). How do the (effective) pressure gradient $\vec{\nabla}\mathcal{P}_{\text{eff.}}$ and flow velocity stand relative to each other?

36. Dimensionless equations of motion for sea surface waves

This exercise is partly a continuation of the May 26 lecture on linear sea surface waves, which you should check if you are not sure of the notations employed.

The equations of motion governing gravity waves at the free surface of an incompressible perfect liquid (ocean/sea water) in a gravity field $-gz\,\vec{e}_z$ are

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0, \tag{2a}$$

$$\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(t,\vec{r}) - g\,\vec{e}_z, \tag{2b}$$

with the boundary conditions $\mathsf{v}_z(t,x,z\!=\!0)=0$ at the sea bottom;

$$\mathsf{v}_z\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \frac{\partial\delta h(t,x)}{\partial t} + \mathsf{v}_x(t,\vec{r})\frac{\partial\delta h(t,x)}{\partial x} \tag{2c}$$

at the free surface, situated at $z = h_0 + \delta h(t,x)$; and a uniform pressure at that same free surface, which may be re-expressed as

$$\mathcal{P}\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \rho g\,\delta h(t,x) + \mathcal{P}_0 \tag{2d}$$

with $\mathcal{P}_0$ a constant whose precise value is irrelevant. As in the lecture (May 26), the problem is assumed to be two-dimensional.

i. We introduce characteristic scales for various quantities: δh_c for the amplitude of the surface deformation; L_c for lengths along the horizontal direction x; and t_c for durations—in practice, the "good" choice would be $t_c = L_c/c_s$ with c_s the speed of sound, yet this is irrelevant here. With their help, we define dimensionless variables

$$t^* \equiv \frac{t}{t_c}, \quad x^* \equiv \frac{x}{L_c}, \quad z^* \equiv \frac{z}{L_c},$$

and fields:

$$\delta h^* \equiv \frac{\delta h}{\delta h_c}, \quad \mathsf{v}_x^* \equiv \frac{\mathsf{v}_x}{\delta h_c/t_c}, \quad \mathsf{v}_z^* \equiv \frac{\mathsf{v}_z}{\delta h_c/t_c}, \quad \mathcal{P}^* \equiv \frac{\mathcal{P}-\mathcal{P}_0}{\rho\,\delta h_c L_c/t_c^2}.$$

Considering the latter as functions of the reduced variables t^*, x^*, z^*, rewrite the equations (2a)–(2d), making use of the dimensionless numbers

$$\mathrm{Fr} \equiv \frac{\sqrt{L_c/g}}{t_c}, \quad \varepsilon \equiv \frac{\delta h_c}{L_c}, \quad \delta \equiv \frac{h_0}{L_c}.$$

What does the parameter ε control (mathematically)? and the parameter δ (physically)?

ii. Assuming that the flow is irrotational, show that you can combine some of the dimensionless equations found in question **i.** into

$$\frac{\partial\mathsf{v}_x^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^*\frac{\partial\mathsf{v}_x^*}{\partial x^*} + \mathsf{v}_z^*\frac{\partial\mathsf{v}_z^*}{\partial x^*}\right) + \frac{1}{\mathrm{Fr}^2}\frac{\partial\delta h^*}{\partial x^*} = 0.$$

The various equations you have obtained in this exercise will be exploited later in the lecture, to derive the *Korteweg–de Vries equation*, which governs in a specific limit the evolution of the function $\phi(t^*,x^*) \equiv \delta h^*(t^*,x^*)/\delta$, i.e. the profile of the free water surface.

Tutorial sheet 13

Discussion topic: Turbulence in fluids: what is it? why does it require a Reynolds number larger than some critical value to develop? In fully developed turbulence, what are the mean flow, the fluctuating flow, the Reynolds stress tensor, the energy cascade?

37. Equations of motion of fully developed turbulence

In the lectures of June 30 and July 2, the detailed derivations of a few equations were left aside. The purpose of this exercise is to fill the gaps.

The velocity field resp. pressure for an incompressible turbulent flow is split into an average and a fluctuating part as

$$\vec{\mathsf{v}}(t,\vec{r}) = \overline{\vec{\mathsf{v}}}(t,\vec{r}) + \vec{\mathsf{v}}'(t,\vec{r}) \quad \text{resp.} \quad \mathcal{P}(t,\vec{r}) = \overline{\mathcal{P}}(t,\vec{r}) + \mathcal{P}'(t,\vec{r}),$$

where the motion with $\overline{\vec{\mathsf{v}}}$, $\overline{\mathcal{P}}$ is referred to as "mean flow". For the sake of simplicity, a system of Cartesian coordinates is being assumed—the components of the gradient thus involve partial derivatives, instead of the more general covariant derivatives. Throughout the exercise, Einstein's summation convention over repeated indices is used.

i. Dynamics of the mean flow

Check that the incompressible Navier–Stokes equation obeyed by $\vec{\mathsf{v}}$ and $\mathcal{P}$ leads for the mean-flow quantities to the equation

$$\frac{\partial\overline{\mathsf{v}^i}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{\mathsf{v}^i} = -\frac{1}{\rho}\frac{\partial\overline{\mathcal{P}}}{\partial x_i} - \frac{\partial\overline{\mathsf{v}'^i\mathsf{v}'^j}}{\partial x^j} + \nu\triangle\overline{\mathsf{v}^i}. \tag{1}$$

Show that this gives for the kinetic energy per unit mass $\overline{k} \equiv \frac{1}{2}\overline{\vec{\mathsf{v}}}^2$ associated with the mean flow the evolution equation

$$\frac{\partial\overline{k}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{k} = -\frac{\partial}{\partial x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\,\overline{\mathsf{v}^j} + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathsf{v}_i}\right] + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathbf{S}_{ij}} \tag{2}$$

with $\overline{\mathbf{S}^{ij}} \equiv \dfrac{1}{2}\left(\dfrac{\partial\overline{\mathsf{v}^i}}{\partial x_j} + \dfrac{\partial\overline{\mathsf{v}^j}}{\partial x_i} - \dfrac{2}{3}g^{ij}\vec{\nabla}\cdot\overline{\vec{\mathsf{v}}}\right)$ the components of the (mean) rate-of-shear tensor.

ii. Dynamics of the fluctuating velocity field

Check that the incompressible Navier–Stokes equation obeyed by $\vec{\mathsf{v}}$ and $\mathcal{P}$ and Eq. (1) for the mean-flow quantities lead for the fluctuating part of the velocity to the equation

$$\frac{\partial\mathsf{v}'^i}{\partial t} + \big(\vec{\mathsf{v}}'\cdot\vec{\nabla}\big)\overline{\mathsf{v}^i} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\mathsf{v}'^i + \frac{\partial}{\partial x^j}\Big(\mathsf{v}'^i\mathsf{v}'^j - \overline{\mathsf{v}'^i\mathsf{v}'^j}\Big) = -\frac{1}{\rho}\frac{\partial\mathcal{P}'}{\partial x_i} + \nu\triangle\mathsf{v}'^i. \tag{3}$$

Show that this gives for the mean kinetic energy $\overline{k'} \equiv \frac{1}{2}\overline{\vec{\mathsf{v}}'^2}$ associated with the flow fluctuations the evolution equation

$$\frac{\partial\overline{k'}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{k'} = -\frac{\partial}{\partial x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}'\mathsf{v}'^j} + \frac{1}{2}\overline{\mathsf{v}'_i\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathsf{v}'_i\mathbf{S}'^{ij}}\right] - \overline{\mathsf{v}'^i\mathsf{v}'^j}\,\overline{\mathbf{S}_{ij}} - 2\nu\overline{\mathbf{S}'^{ij}\mathbf{S}'_{ij}} \tag{4}$$

with $\mathbf{S}'^{ij} \equiv \dfrac{1}{2}\left(\dfrac{\partial\mathsf{v}'^i}{\partial x_j} + \dfrac{\partial\mathsf{v}'^j}{\partial x_i} - \dfrac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{\mathsf{v}}'\right)$ the components of the fluctuating rate-of-shear tensor.

38. A mathematical model to reproduce some features of fully developed turbulence

While trying to solve the problem of turbulence in (incompressible) fluids, Burgers[1] wrote down a system of simpler equations—a toy mathematical model—that share a few features of the dynamical equations (1), (3), namely

$$\frac{\mathrm{d}\bar{\mathsf{v}}(t)}{\mathrm{d}t} = \mathcal{P} - \mathsf{v}'(t)^2 - \nu\,\bar{\mathsf{v}}(t), \tag{5a}$$

$$\frac{\mathrm{d}\mathsf{v}'(t)}{\mathrm{d}t} = \bar{\mathsf{v}}(t)\mathsf{v}'(t) - \nu\,\mathsf{v}'(t), \tag{5b}$$

with $\bar{\mathsf{v}}$, v' two unknown functions, while ν is a parameter and $\mathcal{P}$ a constant. In these equations, all quantities are dimensionless.
The questions **i.**, **ii.**, **iii.**, **iv.** are to a very large extent independent from each other.

i. Enumerate the similarities between Burgers' set of equations and the "true" ones (1), (3). That is, identify the physical content of each term in Eqs. (5), and recognize how key mathematical features of the fluid dynamical equations are seemingly reproduced—while others are obviously not, which may deserve a discussion as well.

ii. Viewing $\bar{\mathsf{v}}$, v' as velocities, write down the differential equation governing the evolution of the sum of the associated kinetic energies (per unit mass...). Note that the terms which you obtain have a straightforward physical interpretation, which smoothly matches those found in question **i.**

iii. "Laminar" solution

a) Show that equations (5) admit a set of stationary solutions with a finite "mean flow velocity" $\bar{\mathsf{v}} = \bar{\mathsf{v}}_0$ and a vanishing "fluctuating velocity" v'.

b) Check that these solutions are stable as long as $\mathcal{P} < \nu^2$. That is, any perturbation ($\delta\bar{\mathsf{v}}$, $\delta\mathsf{v}'$) yielding total velocities $\bar{\mathsf{v}}(t) = \bar{\mathsf{v}}_0 + \delta\bar{\mathsf{v}}(t)$, $\mathsf{v}'(t) = \delta\mathsf{v}'(t)$ will be exponentially damped. On the other hand, the solution ($\bar{\mathsf{v}} = \bar{\mathsf{v}}_0$, $\mathsf{v}' = 0$) is unstable for $\mathcal{P} > \nu^2$.

iv. "Turbulent" solution

Let us now assume $\mathcal{P} > \nu^2$.

a) Show that equations (5) now admit two sets of stationary solutions, both involving a finite mean flow velocity $\bar{\mathsf{v}}$—the same for both sets—and a finite fluctuating velocity $\mathsf{v}' = \pm\mathsf{v}'_0$.

b) Show that both solutions are stable for $\mathcal{P} > \nu^2$.
Hint: You should have to distinguish two cases, namely $\nu < \mathcal{P} \leq \frac{9}{8}\nu^2$ and $\mathcal{P} > \frac{9}{8}\nu^2$.

The appearance of several regimes—one laminar ($\mathsf{v}' = 0$), the other turbulent ($\mathsf{v}' \neq 0$)—depending on the value of a parameter is reminiscent of the onset of turbulence above a geometry-dependent given Reynolds number in the real fluid dynamical case: in that respect, Burgers' toy model reproduces an important feature of the true equations. On the other hand, the existence of two competing turbulent solutions above the critical parameter value is an over-simplification of the real turbulent motion.

[1] J. Burgers, 1895–1981

Tutorial sheet 14

summary list of discussion topics

- Which idealizations underlie the description of a macroscopic many-body system as a continuous medium? What is the Knudsen number?
- How is local thermodynamic equilibrium defined?
- What are the Lagrangian and Eulerian descriptions of fluid motion?
- What is mechanical stress? the (Cauchy) stress tensor? How is a fluid defined?
- What are the strain rate tensor, the rotation rate tensor, the vorticity vector? How do they come about and what do they measure?
- What do the following characterizations of flows mean?
  - compressible/incompressible
  - vorticity-free
  - laminar
  - steady
  - isentropic
  - subsonic/supersonic. How is the Mach number defined?
- What is the Reynolds transport theorem (and its utility)?
- Which quantities are used to describe the motion of a fluid?
- What are the basic equations governing the dynamics of perfect fluids? What are the boundary conditions?
- What is the Bernoulli equation? Which examples of application do you know?
- What is Kelvin's circulation theorem? What does it imply for the vorticity?
- What is a potential flow? What are the corresponding equations of motion?
- What is a sound wave? How do you derive the corresponding equation of motion? How is the speed of sound defined? What happens when the amplitude of the wave becomes large?
- What are the fundamental equations governing the dynamics of non-relativistic Newtonian fluids? What are the boundary conditions?
- Do you know simple examples of steady flows of non-relativistic Newtonian fluids?
- How can one reformulate the Navier–Stokes equation in dimensionless form? What is dynamical similarity?
- How is the Reynolds number defined?
- Which modifications due to viscosity affect the dynamics of vorticity? of sound waves?

- Turbulence in fluids: what is it? why does it require a Reynolds number larger than some critical value to develop?
- In fully developed turbulence, what are the mean flow, the fluctuating flow, the Reynolds stress tensor, the energy cascade?
- Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects plays a role?
- What are the fundamental equations of the dynamics of a relativistic fluid?
- What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?
- In the case of a flowing dissipative relativistic fluid, what is the Eckart frame? the Landau frame?
- What do the denominations "first-order", "second-order" dissipative fluid dynamics stand for?