Tutorial sheet 5

The exercise marked with a star is homework.

Discussion topics:

- Give the basic equations governing the dynamics of perfect fluids.
- What is the Bernoulli equation? Give some examples of application.

*12. Simplified model of star

In an oversimplified approach, one may model a star as a sphere of fluid—a plasma—with uniform mass density ρ. This fluid is in mechanical equilibrium under the influence of pressure $\mathcal{P}$ and gravity. Throughout this exercise, the rotation of the star is neglected.

i. Determine the gravitational field at a distance r from the center of the star.

ii. Assuming that the pressure only depend on r, write down the equation expressing the mechanical equilibrium of the fluid. Determine the resulting function $\mathcal{P}(r)$. Compute the pressure at the star center as function of the mass M and radius R of the star. Calculate the numerical value of this pressure for $M = 2 \times 10^{30}$ kg (solar mass) and $R = 7 \times 10^{8}$ m (solar radius).

iii. The matter constituting the star is assumed to be an electrically neutral mixture of hydrogen nuclei and electrons. Show that the order of magnitude of the total particle number density of that plasma is $n \approx 2\rho/m_p$, with m_p the proton mass. Estimate the temperature at the center of the sun.

Hint: $m_p = 1.6 \times 10^{-27}$ kg; $k_{\mathrm{B}} = 1.38 \times 10^{-23}$ $\mathrm{J \cdot K^{-1}}$.

13. Rotating fluid in a uniform gravitational potential

Consider a perfect fluid contained in a straight cylindrical vessel which rotates with constant angular velocity $\vec{\Omega} = \Omega\,\vec{e}_3$ about its vertical axis, the whole system being placed in a uniform gravitational field $-g\,\vec{e}_3$. Assuming that the fluid rotates with the same angular velocity and that its motion is incompressible, determine the shape of the free surface of the fluid.

Hint: Despite the geometry, working with Cartesian coordinates is quite straightforward. At the free surface, the fluid pressure is constant (it equals the atmospheric pressure).

14. Stationary vortex

Let $\vec{\omega}(t,\vec{r}) = A\,\delta(x^1)\,\delta(x^2)\,\vec{e}_3$ be the vorticity field in a fluid, with A a real constant and $\{x^i\}$ Cartesian coordinates. Determine the corresponding flow velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

Hint: You should invoke symmetry arguments and Stokes' theorem. A useful formal analogy is provided by the Maxwell–Ampère equation of magnetostatics.