Tutorial sheet 4

The exercise marked with a star is homework.

Discussion topic: What is a perfect fluid? a Newtonian fluid? What are the basic equations governing their respective motions?

*10. General momentum and energy equations in a fluid

In this exercise, the roman indices $(i,j\dots)$ denote the coordinates of vectors and tensors in a Cartesian coordinate system, and ∂_i denotes the partial derivative with respect to the i-th coordinate of the position vector $\vec{r}$ in that system. For brevity, the variables $(t,\vec{r})$ of the fields are omitted, and Einstein's summation convention over repeated indices is used.

i. Momentum equation

Check that the Euler and Navier–Stokes equations are special cases of the more general equation

$$\rho\frac{\mathrm{D}\mathsf{v}^j}{\mathrm{D}t} = \partial_i\boldsymbol{\sigma}^{ij} + f_V^j, \tag{1}$$

with $\boldsymbol{\sigma}^{ij}$ the components of Cauchy's stress tensor and f_V^i those of the external volume forces acting on the fluid. If needed, you can find the so-called "constitutive equations" defining $\boldsymbol{\sigma}^{ij}$ for a perfect and a Newtonian fluid in the lecture notes.

ii. Energy equations

a) Check the identity

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\vec{\mathsf{v}}^2 + \frac{e}{\rho}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{\mathsf{v}}^2 + e\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\vec{\mathsf{v}}^2 + e\right)\vec{\mathsf{v}}\right]. \tag{2}$$

Note that e/ρ is the specific internal energy.

Assuming that the volume forces derive from a time-independent potential energy, $\vec{f}_V = -\rho\vec{\nabla}\Phi$, show the identity

$$\frac{\partial(\rho\Phi)}{\partial t} + \vec{\nabla}\cdot\big(\rho\Phi\vec{\mathsf{v}}\big) = -\vec{f}_V\cdot\vec{\mathsf{v}}. \tag{3}$$

What does the term on the right-hand side represent?

b) Using the identities of the previous question, show that the energy equations given for perfect and Newtonian fluids in the lecture are special cases of the more general equation

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\vec{\mathsf{v}}^2 + \frac{e}{\rho}\right) = -\vec{\nabla}\cdot\vec{\jmath}_Q + \partial_i\big(\boldsymbol{\sigma}^{ij}\mathsf{v}_j\big) + \vec{f}_V\cdot\vec{\mathsf{v}}, \tag{4}$$

where $\vec{\jmath}_Q$ denotes the heat flux density.

c) Part of equation (4) is purely mechanical: show that the momentum equation (1) leads to

$$\rho\frac{\mathrm{D}(\frac{1}{2}\vec{\mathsf{v}}^2)}{\mathrm{D}t} = \mathsf{v}_j\partial_i\boldsymbol{\sigma}^{ij} + \vec{f}_V\cdot\vec{\mathsf{v}}, \tag{5}$$

Deduce therefrom the equation governing the rate of change of the specific internal energy. Which form does this equation for the material derivative of e/ρ take in the case of a perfect fluid? Can you deduce an equation for $\mathrm{D}e/\mathrm{D}t$ in perfect fluids, involving the *enthalpy density* $w \equiv e + \mathcal{P}$?

The general equation for e/ρ (or e) can then be used, together with thermodynamic identities, to derive an equation governing the rate of change of the entropy — but not here.

11. Harmonic oscillations in a fluid

Consider the idealized case of a fluid with uniform, yet time dependent mass density: $\rho(t)$. The fluid velocity field is assumed to be spherically symmetric and with a harmonic dependence on time: $\vec{\mathsf{v}}(t,\vec{r}) = \mathsf{v}(r)\cos(\omega_0 t)\,\vec{\mathrm{e}}_r$, where $\vec{\mathrm{e}}_r \equiv \vec{r}/|\vec{r}| \equiv \vec{r}/r$ and ω_0 a given angular frequency.

Determine the mass density $\rho(t)$ and the dependence $\mathsf{v}(r)$.

Hint: In spherical coordinates, the divergence of a purely radial vector field $\vec{V}(\vec{r}) = V_r(r)\,\vec{\mathrm{e}}_r$ is

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\big[r^2 V_r(r)\big].$$