Tutorial sheet 3

The exercise marked with a star is homework.

Discussion topics:

- What is the Reynolds transport theorem (and its utility)?
- What is a perfect fluid? a Newtonian fluid?

*7. A flow with cylindrical symmetry: pointlike source

In this exercise and the following one, we use a system of cylindrical coordinates (r, θ, z) with *unit* basis vectors $(\vec{u}_r, \vec{u}_\theta, \vec{u}_z)$. Accordingly, the divergence of a vector field[1] $\vec{V}(\vec{r}) = V^r\vec{u}_r + V^\theta\vec{u}_\theta + V^z\vec{u}_z$ is given by

$$\vec{\nabla} \cdot \vec{V}(\vec{r}) = \frac{1}{r}\frac{\partial(rV^r)}{\partial r} + \frac{1}{r}\frac{\partial V^\theta}{\partial \theta} + \frac{\partial V^z}{\partial z}.$$

Consider the fluid motion defined for $r \neq 0$ by the velocity field

$$\mathsf{v}^r(t, \vec{r}) = \frac{f(t)}{r}, \quad \mathsf{v}^\theta(t, \vec{r}) = 0, \quad \mathsf{v}^z(t, \vec{r}) = 0,$$

with f some scalar function.

a) Compute the volume expansion rate and the vorticity vector.

b) Mathematically, the velocity field is singular at $r = 0$. Thinking of the velocity profile, what do you have *physically* at that point if $f(t) > 0$? if $f(t) < 0$?

8. Pointlike vortex

Consider now the fluid motion defined for $r \neq 0$ by the velocity field

$$\vec{\mathsf{v}}(t, \vec{r}) = \frac{\Gamma}{2\pi r}\vec{u}_\theta, \quad \Gamma \in \mathbb{R}.$$

i. Give the corresponding volume expansion rate and vorticity vector. Compute the *circulation* of the velocity field along a closed curve circling the z-axis. For which physical phenomenon could this motion be a (very crude!) model?

ii. The velocity fields of exercise **7** — assuming that $f(t)$ is time-independent — and the present exercise are analogous to the electrical or magnetic fields created by simple (stationary) distributions of electric charges or currents. Do you see which?

9. Symmetry of the stress tensor

Let $\boldsymbol{\sigma}_{ij} = -\mathbf{T}_{ij}$ denote the Cartesian components of the stress tensor in a continuous medium. Consider an infinitesimal cube of medium, whose edges (length $\mathrm{d}\ell$) are parallel to the axes of the coordinate system.

i. Explain why the k-th component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the continuous medium obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\mathbf{T}_{ij}(\mathrm{d}\ell)^3$, with ϵ_{ijk} the usual Levi-Civita symbol.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the medium mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?

[1] For the sake of brevity the dependence of V^r, V^θ, V^z and the basis vectors $\vec{u}_r, \vec{u}_\theta$ on the position $\vec{r}$ is not denoted.