Tutorial sheet 13

The exercise marked with a star is homework.

*34. Speed of sound in ultrarelativistic matter

Consider a perfect fluid with the usual energy-momentum tensor. $T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (\epsilon + \mathcal{P})u^\mu u^\nu/c^2$. It is assumed that there is no conserved quantum number relevant for thermodynamics, so that the energy density in the local rest frame ϵ is function of a single thermodynamic variable, for instance $\epsilon = \epsilon(\mathcal{P})$. Throughout the exercise, Minkowski coordinates are used.

A background "flow" with uniform local-rest-frame energy density and pressure ϵ_0 and $\mathcal{P}_0$ is submitted to a small perturbation resulting in $\epsilon = \epsilon_0 + \delta\epsilon$, $\mathcal{P} = \mathcal{P}_0 + \delta\mathcal{P}$, and $\vec{\mathrm{v}} = \vec{0} + \delta\vec{\mathrm{v}}$.

i. Starting from the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, show that linearization to first order in the perturbations leads to the two equations of motion $\partial_t\delta\epsilon = -(\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\delta\vec{\mathrm{v}}$ and $(\epsilon_0 + \mathcal{P}_0)\partial_t\delta\vec{\mathrm{v}} = -c^2\vec{\nabla}\delta\mathcal{P}$.

ii. Show that the speed of sound is given by the expression $c_s^2 = \dfrac{c^2}{\mathrm{d}\epsilon/\mathrm{d}\mathcal{P}}$.

iii. Compute c_s for a fluid obeying the Stefan–Boltzmann law[1] $\mathcal{P} = \dfrac{g\pi^2}{90}\dfrac{(k_\mathrm{B}T)^4}{(\hbar c)^3}$, with g the number of degrees of freedom (e.g. $g = 2$ for blackbody radiation).

Hint: You may find the Gibbs–Duhem relation useful...

[1]This is a good opportunity to refresh your knowledge on the statistical physics of relativistic systems. Can you give a physical argument why quantum effects always play a role in such systems, as signaled by the presence of $\hbar$ in the equation of state?