Tutorial sheet 12

The exercises marked with a star are homework.

Discussion topics:

– Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects play a role?
– What are the fundamental equations of the dynamics of a relativistic fluid? What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?

*30. Energy-momentum tensor

Let $\mathcal{R}$ denote a fixed reference frame. Consider a perfect fluid whose local rest frame at a point x moves with velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}$. Show with the help of a Lorentz transformation that the Minkowski components of the energy-momentum tensor of the fluid at x are given to order $\mathcal{O}(|\vec{\mathsf{v}}|/c)$ by

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^1}{c} & (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^2}{c} & (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^3}{c} \\ (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^1}{c} & \mathcal{P} & 0 & 0 \\ (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^2}{c} & 0 & \mathcal{P} & 0 \\ (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^3}{c} & 0 & 0 & \mathcal{P} \end{pmatrix},$$

where for the sake of brevity the x-dependence of the various fields is omitted. Check the compatibility of this result with the general formula for $T^{\mu\nu}$ given in the lecture.

*31. A family of solutions of the dynamical equations for perfect relativistic fluids

Let $\{x^\mu\}$ denote Minkowski coordinates and $\tau^2 \equiv -x^\mu x_\mu$, where the "mostly plus" metric is used. Show that the following four-velocity, pressure and charge density constitute a solution of the equations describing the motion of a perfect relativistic fluid with equation of state $\mathcal{P} = K\varepsilon$ and a single conserved charge:

$$u^\mu(\mathsf{x}) = \frac{x^\mu}{\tau} \quad , \quad \mathcal{P}(\mathsf{x}) = \mathcal{P}_0\left(\frac{\tau_0}{\tau}\right)^{3(1+K)} \quad , \quad n(\mathsf{x}) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}\big(\sigma(\mathsf{x})\big), \tag{1}$$

with τ_0, $\mathcal{P}_0$, n_0 arbitrary constants and $\mathcal{N}$ an arbitrary function of a single argument, while σ is a function of spacetime coordinates with vanishing comoving derivative: $u^\mu\partial_\mu\sigma(\mathsf{x}) = 0$.

32. Equations of motion of a perfect relativistic fluid

In this exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

Hint: If the covariant derivatives d_μ upset you, assume you have chosen Minkowski coordinates, in which $\mathrm{d}_\mu = \partial_\mu$.

i. Check that the tensor with components $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ defines a projector on the subspace orthogonal to the 4-velocity.
Denoting by d_μ the components of the (covariant) 4-gradient, we define $\nabla^\nu \equiv \Delta^{\mu\nu}\mathrm{d}_\mu$. Can you see the rationale behind this notation?

ii. Show that the energy-momentum conservation equation for a perfect fluid is equivalent to the two equations

$$u^\mu \mathrm{d}_\mu\epsilon + (\epsilon+\mathcal{P})\mathrm{d}_\mu u^\mu = 0 \quad \text{and} \quad (\epsilon+\mathcal{P})u^\mu\mathrm{d}_\mu u^\nu + \nabla^\nu\mathcal{P} = 0. \tag{2}$$

Which known equation does the second one evoke?

33. Vorticity in a perfect relativistic fluid

Consider the *kinematic vorticity tensor* defined by its components

$$\omega_{\mu\nu} \equiv \frac{1}{2}\Delta^{\alpha}_{\ \mu}\Delta^{\beta}_{\ \nu}\big(\mathrm{d}_\beta u_\alpha - \mathrm{d}_\alpha u_\beta\big), \tag{3}$$

where $\Delta^{\alpha}_{\ \mu} \equiv g^{\alpha}_{\ \mu} + u^\alpha u_\mu$ (see exercise **32.i.**).

a) Why is no calculation necessary to prove the identity $\Delta^{\mu\nu}\omega_{\mu\nu} = 0$? Show the identity

$$\omega_{\mu\nu} = \frac{1}{2}\big(\mathrm{d}_\nu u_\mu - \mathrm{d}_\mu u_\nu + a_\mu u_\nu - a_\nu u_\mu\big), \tag{4}$$

where $a^\mu \equiv u^\nu \mathrm{d}_\nu u^\mu$.

b) Define a four-vector by[1] $\omega^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\omega_{\rho\sigma}u_\nu$. What are its components in the local rest frame? What do you recognize?

c) Show that if $a^\mu = 0$, then the vorticity 4-vector obeys the evolution equation

$$u^\nu \mathrm{d}_\nu \omega^\mu = -\frac{2}{3}(\mathrm{d}_\rho u^\rho)\omega^\mu + S^{\mu}_{\ \nu}\omega^\nu, \tag{5}$$

where $S^{\mu\nu}$ is the rate-of-shear tensor.

[1] The convention $\epsilon^{0123} = -1$ is used.