Tutorial sheet 11

The exercise marked with a star is homework.

Discussion topics:

– Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects play a role?
– What are the fundamental equations of the dynamics of a relativistic fluid? What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?

*28. (1+1)-dimensional relativistic motion

Consider a (1+1)-dimensional relativistic motion along a direction denoted as z, where the denomination "1+1" stands for one time and one spatial dimension. Throughout the exercise, the other two spatial directions play no role and the corresponding variables x, y are totally omitted. In addition, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein's summation convention over repeated indices.

To describe the physics, one may naturally use Minkowski coordinates $(x^0, x^3) = (t, z)$, with corresponding derivatives $(\partial_0, \partial_3) = (\partial/\partial t, \partial/\partial z)$. If there is a high-velocity motion in the z-direction, a better choice might be to use the *proper time* τ and *spatial rapidity* ς such that[1]

$$x^{0'} \equiv \tau \equiv \sqrt{t^2 - z^2}, \quad x^{3'} \equiv \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \quad \text{where } |z| \leq t. \tag{1}$$

The partial derivatives with respect to these new coordinates will be denoted $(\partial_{0'}, \partial_{3'}) = (\partial/\partial\tau, \partial/\partial\varsigma)$.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau \cosh\varsigma, \quad z = \tau \sinh\varsigma. \tag{2}$$

(*Hint:* Recognize $\frac{1}{2} \log \frac{1+u}{1-u}$).

ii. In a change of coordinates $\{x^\mu\} \to \{x^{\mu'}\}$, the contravariant components V^μ of a 4-vector transform according to $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$ (with summation over ν!) where $\Lambda^{\mu'}{}_\nu \equiv \partial x^{\mu'}/\partial x^\nu$.

Compute first from Eq. (2) the matrix elements $\Lambda^\nu{}_{\mu'} \equiv \partial x^\nu/\partial x^{\mu'}$ (with $\nu \in \{0, 3\}$, $\mu' \in \{0', 3'\}$) of the inverse transformation $\{V^{\mu'}\} \to \{V^\mu\}$. Inverting the 2×2-matrix you thus found, deduce the following relationship between the components of the 4-vector in the two coordinate systems

$$\begin{cases} V^{0'} = \cosh\varsigma\, V^0 - \sinh\varsigma\, V^3 \\ V^{3'} = -\dfrac{1}{\tau} \sinh\varsigma\, V^0 + \dfrac{1}{\tau} \cosh\varsigma\, V^3. \end{cases} \tag{3}$$

iii. Using the relation $\partial_\nu = \Lambda^{\mu'}{}_\nu \partial_{\mu'}$ and the matrix elements $\{\Lambda^{\mu'}{}_\nu\}$ you found in **ii.**—and which can be read off Eq. (3)—, express the "4-divergence" $\partial_\nu V^\nu$ of a 4-vector field V^ν in terms of the partial derivatives $\partial_{\mu'}$ and the components $V^{\mu'}$ in the (τ, ς)-system.

You should find a result that does not equal $\partial_{\mu'} V^{\mu'} = \partial_\tau V^\tau + \partial_\varsigma V^\varsigma$, which is why in the lecture notes the notation $\mathrm{d}_{\mu'} V^{\mu'}$ is used for the 4-divergence in an arbitrary coordinate system.

iv. Draw on a spacetime diagram—with t on the vertical axis and z on the horizontal axis—the lines of constant τ and those of constant ς.

Remark: The coordinates (τ, ς) are sometimes called *Milne coordinates*.

[1] ς = `\varsigma` is the word-final form for the lower case sigma, not to be confused with ζ (zeta).

29. Quantum number conservation

Consider a 4-current with components $N^\mu(\mathsf{x})$ obeying the continuity equation $\partial_\mu N^\mu(\mathsf{x}) = 0$, where the $\{x^\mu\}$ hidden in the notation ∂_μ are Minkowski coordinates. Show that the quantity

$$\mathcal{N} = \frac{1}{c}\int N^0(\mathsf{x})\,\mathrm{d}^3\vec{r}$$

is a Lorentz scalar, by convincing yourself first that it can be rewritten in the form

$$\mathcal{N} = \frac{1}{c}\int_{x^0=\text{const.}} N^\mu(\mathsf{x})\,\mathrm{d}^3\sigma_\mu, \qquad (4)$$

where $\mathrm{d}^3\sigma_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ is a 4-vector, with $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ the antisymmetric 4-tensor defined by

$$\mathrm{d}^3\mathcal{V}^{012} = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2, \quad \mathrm{d}^3\mathcal{V}^{021} = -\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^1, \quad \text{etc.}$$

and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric Levi–Civita symbol with the convention $\epsilon_{0123} = +1$, such that $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ represents the 3-dimensional hypersurface element in Minkowski space.