Tutorial sheet 10

The exercise marked with a star is homework.

Discussion topic: Turbulence in fluids: what is it? Why does it require a Reynolds number larger than some critical value to develop? In fully developed turbulence, what are the mean flow, the fluctuating flow, the Reynolds stress tensor, the energy cascade?

*26. A mathematical model to reproduce some features of fully developed turbulence

While trying to solve the problem of incompressible turbulence in fluids, Burgers (again!) wrote down a system of simpler equations—a toy mathematical model—that share a few features of the dynamical equations governing the mean flow and the flow fluctuations, namely

$$\frac{\mathrm{d}\bar{\mathsf{v}}(t)}{\mathrm{d}t} = \mathcal{P} - \mathsf{v}'(t)^2 - \nu\,\bar{\mathsf{v}}(t), \tag{1a}$$

$$\frac{\mathrm{d}\mathsf{v}'(t)}{\mathrm{d}t} = \bar{\mathsf{v}}(t)\mathsf{v}'(t) - \nu\,\mathsf{v}'(t), \tag{1b}$$

with $\bar{\mathsf{v}}$, v' two unknown functions, while ν is a parameter and $\mathcal{P}$ a constant. In these equations, all quantities (including t) are dimensionless and real.
The questions **i.**, **ii.**, **iii.**, **iv.** are to a very large extent independent from each other.

i. Enumerate the similarities between Burgers' set of equations and the "true" ones given in the lecture. That is, identify the physical content of each term in Eqs. (1), and recognize how key mathematical features of the fluid dynamical equations are seemingly reproduced—while others are obviously not, which may deserve a discussion as well.

ii. Viewing $\bar{\mathsf{v}}$, v' as velocities, write down the differential equation governing the evolution of the sum of the associated kinetic energies (per unit mass…). Note that the terms which you obtain have a straightforward physical interpretation, which smoothly matches those found in question **i.**

iii. "Laminar" solution

a) Show that equations (1) admit a set of stationary solutions with a finite "mean flow velocity" $\bar{\mathsf{v}} = \bar{\mathsf{v}}_0$ and a vanishing "fluctuating velocity" v'.

b) Check that these solutions are stable as long as $\mathcal{P} < \nu^2$. That is, any perturbation $(\delta\bar{\mathsf{v}}, \delta\mathsf{v}')$ yielding total velocities $\bar{\mathsf{v}}(t) = \bar{\mathsf{v}}_0 + \delta\bar{\mathsf{v}}(t)$, $\mathsf{v}'(t) = \delta\mathsf{v}'(t)$ will be exponentially damped. On the other hand, the solution $(\bar{\mathsf{v}} = \bar{\mathsf{v}}_0,\ \mathsf{v}' = 0)$ is unstable for $\mathcal{P} > \nu^2$.

iv. "Turbulent" solution
Let us now assume $\mathcal{P} > \nu^2$.

a) Show that equations (1) now admit two sets of stationary solutions, both involving a finite mean flow velocity $\bar{\mathsf{v}}$—the same for both sets—and a finite fluctuating velocity $\mathsf{v}' = \pm\mathsf{v}'_0$.

b) Show that both solutions are stable for $\mathcal{P} > \nu^2$.
Hint: You should have to distinguish two cases, namely $\nu < \mathcal{P} \le \frac{9}{8}\nu^2$ and $\mathcal{P} > \frac{9}{8}\nu^2$.

The appearance of several regimes—one laminar ($\mathsf{v}' = 0$), the other turbulent ($\mathsf{v}' \neq 0$)—depending on the value of a parameter is reminiscent of the onset of turbulence above a geometry-dependent given Reynolds number in the real fluid dynamical case: in that respect, Burgers' toy model reproduces an important feature of the true equations. On the other hand, the existence of two competing turbulent solutions above the critical parameter value is an over-simplification of the real turbulent motion.

27. Dynamics of the mean flow in fully developed turbulence

The velocity field resp. pressure for an incompressible turbulent flow is split into an average and a fluctuating part as

$$\vec{\mathsf{v}}(t,\vec{r}) = \overline{\vec{\mathsf{v}}}(t,\vec{r}) + \vec{\mathsf{v}}'(t,\vec{r}) \quad \text{resp.} \quad \mathcal{P}(t,\vec{r}) = \overline{\mathcal{P}}(t,\vec{r}) + \mathcal{P}'(t,\vec{r}),$$

where the motion with $\overline{\vec{\mathsf{v}}}$, $\overline{\mathcal{P}}$ is referred to as "mean flow". For the sake of simplicity, a system of Cartesian coordinates is being assumed—the components of the gradient thus involve partial derivatives, instead of the more general covariant derivatives. Throughout the exercise, Einstein's summation convention over repeated indices is used.

Check that the incompressible Navier–Stokes equation obeyed by $\vec{\mathsf{v}}$ and $\mathcal{P}$ leads for the mean-flow quantities to the equation

$$\frac{\partial\overline{\mathsf{v}^i}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{\mathsf{v}^i} = -\frac{1}{\rho}\frac{\partial\overline{\mathcal{P}}}{\partial x_i} - \frac{\partial\overline{\mathsf{v}'^i\mathsf{v}'^j}}{\partial x^j} + \nu\triangle\overline{\mathsf{v}^i}. \tag{2}$$

Show that this gives for the kinetic energy per unit mass $\overline{k} \equiv \frac{1}{2}\overline{\vec{\mathsf{v}}}^2$ associated with the mean flow the evolution equation

$$\frac{\partial\overline{k}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{k} = -\frac{\partial}{\partial x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\overline{\mathsf{v}^j} + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathsf{v}_i}\right] + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathbf{S}_{ij}} \tag{3}$$

with $\overline{\mathbf{S}^{ij}} \equiv \frac{1}{2}\left(\frac{\partial\overline{\mathsf{v}^i}}{\partial x_j} + \frac{\partial\overline{\mathsf{v}^j}}{\partial x_i} - \frac{2}{3}g^{ij}\vec{\nabla}\cdot\overline{\vec{\mathsf{v}}}\right)$ the components of the (mean) rate-of-shear tensor.