Tutorial sheet 1

The exercise marked with a star is probably better suited as homework.

Discussion topic: Which idealizations underlie the description of a macroscopic many-body system as a continuous medium? How is local thermodynamic equilibrium defined?

*1. Gradient, divergence and curl of products

Show the following identities involving the nabla operator $\vec{\nabla}$, where f, f_1 and f_2 denote scalar fields (on $\mathbb{R}^3$) while $\vec{V}$, $\vec{V}_1$ and $\vec{V}_2$ are vector fields.

$$\vec{\nabla}\big[f_1(\vec{r})f_2(\vec{r})\big] = \big[\vec{\nabla}f_1(\vec{r})\big]f_2(\vec{r}) + f_1(\vec{r})\vec{\nabla}f_2(\vec{r}). \tag{1}$$

$$\vec{\nabla}\cdot\big[f(\vec{r})\vec{V}(\vec{r})\big] = \big[\vec{\nabla}f(\vec{r})\big]\cdot\vec{V}(\vec{r}) + f(\vec{r})\vec{\nabla}\cdot\vec{V}(\vec{r}). \tag{2}$$

$$\vec{\nabla}\times\big[f(\vec{r})\vec{V}(\vec{r})\big] = \big[\vec{\nabla}f(\vec{r})\big]\times\vec{V}(\vec{r}) + f(\vec{r})\vec{\nabla}\times\vec{V}(\vec{r}). \tag{3}$$

$$\vec{\nabla}\cdot\big[\vec{V}_1(\vec{r})\times\vec{V}_2(\vec{r})\big] = \vec{V}_2(\vec{r})\cdot\big[\vec{\nabla}\times\vec{V}_1(\vec{r})\big] - \vec{V}_1(\vec{r})\cdot\big[\vec{\nabla}\times\vec{V}_2(\vec{r})\big]. \tag{4}$$

$$\vec{\nabla}\times\big[\vec{V}_1(\vec{r})\times\vec{V}_2(\vec{r})\big] = \big[\vec{\nabla}\cdot\vec{V}_2(\vec{r})\big]\vec{V}_1(\vec{r}) - \big[\vec{\nabla}\cdot\vec{V}_1(\vec{r})\big]\vec{V}_2(\vec{r}) + \big[\vec{V}_2(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_1(\vec{r}) - \big[\vec{V}_1(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_2(\vec{r}). \tag{5}$$

$$\vec{\nabla}\big[\vec{V}_1(\vec{r})\cdot\vec{V}_2(\vec{r})\big] = \vec{V}_1(\vec{r})\times\big[\vec{\nabla}\times\vec{V}_2(\vec{r})\big] + \vec{V}_2(\vec{r})\times\big[\vec{\nabla}\times\vec{V}_1(\vec{r})\big] + \big[\vec{V}_1(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_2(\vec{r}) + \big[\vec{V}_2(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_1(\vec{r}). \tag{6}$$

Hint: You may introduce Cartesian coordinates if you wish.

2. Stationary flow: first example

(This exercise introduces a number of concepts that will only be introduced in later lectures; this should pose you no difficulty.)

Consider the stationary flow defined in the region $x^1 > 0$, $x^2 > 0$ by its velocity field

$$\vec{\mathrm{v}}(t,\vec{r}) = k(-x^1\,\vec{\mathrm{e}}_1 + x^2\,\vec{\mathrm{e}}_2) \tag{7}$$

with k a positive constant, $\{\vec{\mathrm{e}}_i\}$ the basis vectors of a Cartesian coordinate system and $\{x^i\}$ the coordinates of the position vector $\vec{r}$.

i. Vector analysis

a) Compute the divergence $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})$ of the velocity field (7). Check that your result is consistent with the existence of a scalar function $\psi(t,\vec{r})$ (the *stream function*) such that

$$\vec{\mathrm{v}}(t,\vec{r}) = -\vec{\nabla}\times\big[\psi(t,\vec{r})\,\vec{\mathrm{e}}_3\big] \tag{8}$$

and determine $\psi(t,\vec{r})$ — there is an arbitrary additive constant, which you may set equal to zero. What are the lines of constant $\psi(t,\vec{r})$?

b) Compute now the curl $\vec{\nabla}\times\vec{\mathrm{v}}(t,\vec{r})$ and deduce therefrom the existence of a scalar function $\varphi(t,\vec{r})$ (the *velocity potential*) such that

$$\vec{\mathrm{v}}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r}). \tag{9}$$

(*Hint*: remember a theorem you saw in your lectures on classical mechanics and/or electromagnetism.) What are the lines of constant $\varphi(t,\vec{r})$?

ii. Stream lines

Determine the *stream lines* at some arbitrary time t. The latter are by definition lines $\vec{\xi}(\lambda)$ whose tangent is everywhere parallel to the instantaneous velocity field, with λ a parameter along the stream line. That is, they obey the condition

$$\frac{\mathrm{d}\vec{\xi}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\,\vec{\mathrm{v}}(t,\vec{\xi}(\lambda))$$

with $\alpha(\lambda)$ a scalar function, or equivalently

$$\frac{\mathrm{d}\xi^1(\lambda)}{\mathsf{v}^1(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^2(\lambda)}{\mathsf{v}^2(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^3(\lambda)}{\mathsf{v}^3(t,\vec{\xi}(\lambda))},$$

with $\mathrm{d}\xi^i(\lambda)$ the coordinates of the (infinitesimal) tangent vector to the stream line.

3. Wave equation

Consider a scalar field $\phi(t,x)$ which obeys the partial differential equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(t,x) = 0 \tag{10}$$

with initial conditions $\phi(0,x) = \mathrm{e}^{-x^2}$, $\partial_t\phi(0,x) = 0$. Determine the solution $\phi(t,x)$ for $t > 0$.