Tutorial sheet 1

The exercise marked with a star is probably better suited as homework.

Discussion topic: Which idealizations underlie the description of a macroscopic many-body system as a continuous medium? How is local thermodynamic equilibrium defined?

*1. Gradient, divergence and curl of products

Show the following identities involving the nabla operator $\vec{\nabla}$, where f, f_1 and f_2 denote scalar fields (on $\mathbb{R}^3$) while $\vec{V}$, $\vec{V}_1$ and $\vec{V}_2$ are vector fields.

$$\vec{\nabla}\big[f_1(\vec{r})f_2(\vec{r})\big] = \big[\vec{\nabla}f_1(\vec{r})\big]f_2(\vec{r}) + f_1(\vec{r})\vec{\nabla}f_2(\vec{r}). \tag{1}$$

$$\vec{\nabla}\cdot\big[f(\vec{r})\vec{V}(\vec{r})\big] = \big[\vec{\nabla}f(\vec{r})\big]\cdot\vec{V}(\vec{r}) + f(\vec{r})\vec{\nabla}\cdot\vec{V}(\vec{r}). \tag{2}$$

$$\vec{\nabla}\times\big[f(\vec{r})\vec{V}(\vec{r})\big] = \big[\vec{\nabla}f(\vec{r})\big]\times\vec{V}(\vec{r}) + f(\vec{r})\vec{\nabla}\times\vec{V}(\vec{r}). \tag{3}$$

$$\vec{\nabla}\cdot\big[\vec{V}_1(\vec{r})\times\vec{V}_2(\vec{r})\big] = \vec{V}_2(\vec{r})\cdot\big[\vec{\nabla}\times\vec{V}_1(\vec{r})\big] - \vec{V}_1(\vec{r})\cdot\big[\vec{\nabla}\times\vec{V}_2(\vec{r})\big]. \tag{4}$$

$$\vec{\nabla}\times\big[\vec{V}_1(\vec{r})\times\vec{V}_2(\vec{r})\big] = \big[\vec{\nabla}\cdot\vec{V}_2(\vec{r})\big]\vec{V}_1(\vec{r}) - \big[\vec{\nabla}\cdot\vec{V}_1(\vec{r})\big]\vec{V}_2(\vec{r}) + \big[\vec{V}_2(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_1(\vec{r}) - \big[\vec{V}_1(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_2(\vec{r}). \tag{5}$$

$$\vec{\nabla}\big[\vec{V}_1(\vec{r})\cdot\vec{V}_2(\vec{r})\big] = \vec{V}_1(\vec{r})\times\big[\vec{\nabla}\times\vec{V}_2(\vec{r})\big] + \vec{V}_2(\vec{r})\times\big[\vec{\nabla}\times\vec{V}_1(\vec{r})\big] + \big[\vec{V}_1(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_2(\vec{r}) + \big[\vec{V}_2(\vec{r})\cdot\vec{\nabla}\big]\vec{V}_1(\vec{r}). \tag{6}$$

Hint: You may introduce Cartesian coordinates if you wish.

2. Stationary flow: first example

(This exercise introduces a number of concepts that will only be introduced in later lectures; this should pose you no difficulty.)

Consider the stationary flow defined in the region $x^1 > 0$, $x^2 > 0$ by its velocity field

$$\vec{\mathrm{v}}(t,\vec{r}) = k(-x^1\,\vec{\mathrm{e}}_1 + x^2\,\vec{\mathrm{e}}_2) \tag{7}$$

with k a positive constant, $\{\vec{\mathrm{e}}_i\}$ the basis vectors of a Cartesian coordinate system and $\{x^i\}$ the coordinates of the position vector $\vec{r}$.

i. Vector analysis

a) Compute the divergence $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})$ of the velocity field (7). Check that your result is consistent with the existence of a scalar function $\psi(t,\vec{r})$ (the *stream function*) such that

$$\vec{\mathrm{v}}(t,\vec{r}) = -\vec{\nabla}\times\big[\psi(t,\vec{r})\,\vec{\mathrm{e}}_3\big] \tag{8}$$

and determine $\psi(t,\vec{r})$ — there is an arbitrary additive constant, which you may set equal to zero. What are the lines of constant $\psi(t,\vec{r})$?

b) Compute now the curl $\vec{\nabla}\times\vec{\mathrm{v}}(t,\vec{r})$ and deduce therefrom the existence of a scalar function $\varphi(t,\vec{r})$ (the *velocity potential*) such that

$$\vec{\mathrm{v}}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r}). \tag{9}$$

(*Hint*: remember a theorem you saw in your lectures on classical mechanics and/or electromagnetism.) What are the lines of constant $\varphi(t,\vec{r})$?

ii. Stream lines

Determine the *stream lines* at some arbitrary time t. The latter are by definition lines $\vec{\xi}(\lambda)$ whose tangent is everywhere parallel to the instantaneous velocity field, with λ a parameter along the stream line. That is, they obey the condition

$$\frac{\mathrm{d}\vec{\xi}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\,\vec{\mathrm{v}}(t,\vec{\xi}(\lambda))$$

with $\alpha(\lambda)$ a scalar function, or equivalently

$$\frac{\mathrm{d}\xi^1(\lambda)}{\mathsf{v}^1(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^2(\lambda)}{\mathsf{v}^2(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^3(\lambda)}{\mathsf{v}^3(t,\vec{\xi}(\lambda))},$$

with $\mathrm{d}\xi^i(\lambda)$ the coordinates of the (infinitesimal) tangent vector to the stream line.

3. Wave equation

Consider a scalar field $\phi(t,x)$ which obeys the partial differential equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(t,x) = 0 \tag{10}$$

with initial conditions $\phi(0,x) = \mathrm{e}^{-x^2}$, $\partial_t\phi(0,x) = 0$. Determine the solution $\phi(t,x)$ for $t > 0$.

Tutorial sheet 2

The exercise marked with a star is homework.

Discussion topics:

- What are the Lagrangian and Eulerian descriptions? How is a fluid defined?
- What are the strain rate tensor, the rotation rate tensor, and the vorticity vector? How do they come about and what do they measure?

*4. Stationary flow: second example

Consider the fluid flow whose velocity field $\vec{\mathsf{v}}(t,\vec{r})$ has coordinates (in a given Cartesian system)

$$\mathsf{v}^1(t,\vec{r}) = kx^2, \quad \mathsf{v}^2(t,\vec{r}) = kx^1, \quad \mathsf{v}^3(t,\vec{r}) = 0, \tag{1}$$

where k is a positive real number, while x^1, x^2, x^3 are the coordinates of the position vector $\vec{r}$.

i. Determine the stream lines at an arbitrary instant t.

ii. Let X^1, X^2, X^3 denote the coordinates of some arbitrary point M and let t_0 be the real number defined by

$$kt_0 = \begin{cases} -\text{Artanh}(X^2/X^1) & \text{if } |X^1| > |X^2| \\ 0 & \text{if } \ X^1 = \pm X^2 \\ -\text{Artanh}(X^1/X^2) & \text{if } |X^1| < |X^2|. \end{cases}$$

Write down a parameterization $x^1(t)$, $x^2(t)$, $x^3(t)$, in terms of a parameter denoted by t, of the coordinates of *the* stream line $\vec{x}(t)$ going through M such that $\mathrm{d}\vec{x}(t)/\mathrm{d}t$ at any point *equals* the velocity field at that point, and that either $x^1(t) = 0$ or $x^2(t) = 0$ for $t = t_0$.

iii. Viewing $\vec{x}(t)$ as the trajectory of a point—actually, of a fluid particle—, you already know the velocity of that point at time t (do you?). What is its acceleration $\vec{a}(t)$?

iv. Coming back to the velocity field (1), compute first its partial derivative $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$, then the *material derivative*

$$\frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t} \equiv \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r}).$$

Compare $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$ and $\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})/\mathrm{D}t$ with the acceleration of a fluid particle found in question **iii.**

5. Yet another example of motion of a deformable continuous medium

Consider the motion defined in a system of Cartesian coordinates with basis vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ by the velocity field with components

$$\mathsf{v}^1(t,\vec{r}) = f_1(t,x^2), \quad \mathsf{v}^2(t,\vec{r}) = f_2(t,x^1), \quad \mathsf{v}^3(t,\vec{r}) = 0,$$

with f_1, f_2 two continuously differentiable functions.

Compute the strain rate tensor $\mathbf{D}(t,\vec{r})$ for this motion. What is the volume expansion rate? Give the rotation rate tensor $\mathbf{R}(t,\vec{r})$ and the vorticity vector. Under which condition(s) on the functions f_1, f_2 does the motion become irrotational?

6. Isotropy of static pressure

Consider a geometrical point at position $\vec{r}$ in a fluid at rest. The stress vector across every surface element going through this point is normal: $\vec{T}(\vec{r}) = -\mathcal{P}(\vec{r})\,\vec{e}_n$, with $\vec{e}_n$ the unit vector orthogonal to the surface element under consideration. Show that the (hydrostatic) pressure $\mathcal{P}$ is independent of the orientation of $\vec{e}_n$.

Hint: Consider the forces on the faces of an infinitesimal trirectangular tetrahedron.

Tutorial sheet 3

The exercise marked with a star is homework.

Discussion topics:

- What is the Reynolds transport theorem (and its utility)?
- What is a perfect fluid? a Newtonian fluid?

*7. A flow with cylindrical symmetry: pointlike source

In this exercise and the following one, we use a system of cylindrical coordinates (r, θ, z) with *unit* basis vectors $(\vec{u}_r, \vec{u}_\theta, \vec{u}_z)$. Accordingly, the divergence of a vector field[1] $\vec{V}(\vec{r}) = V^r \vec{u}_r + V^\theta \vec{u}_\theta + V^z \vec{u}_z$ is given by

$$\vec{\nabla} \cdot \vec{V}(\vec{r}) = \frac{1}{r}\frac{\partial(rV^r)}{\partial r} + \frac{1}{r}\frac{\partial V^\theta}{\partial \theta} + \frac{\partial V^z}{\partial z}.$$

Consider the fluid motion defined for $r \neq 0$ by the velocity field

$$\mathsf{v}^r(t, \vec{r}) = \frac{f(t)}{r}, \quad \mathsf{v}^\theta(t, \vec{r}) = 0, \quad \mathsf{v}^z(t, \vec{r}) = 0,$$

with f some scalar function.

a) Compute the volume expansion rate and the vorticity vector.

b) Mathematically, the velocity field is singular at $r = 0$. Thinking of the velocity profile, what do you have *physically* at that point if $f(t) > 0$? if $f(t) < 0$?

8. Pointlike vortex

Consider now the fluid motion defined for $r \neq 0$ by the velocity field

$$\vec{\mathsf{v}}(t, \vec{r}) = \frac{\Gamma}{2\pi r}\vec{u}_\theta, \quad \Gamma \in \mathbb{R}.$$

i. Give the corresponding volume expansion rate and vorticity vector. Compute the *circulation* of the velocity field along a closed curve circling the z-axis. For which physical phenomenon could this motion be a (very crude!) model?

ii. The velocity fields of exercise **7** — assuming that $f(t)$ is time-independent — and the present exercise are analogous to the electrical or magnetic fields created by simple (stationary) distributions of electric charges or currents. Do you see which?

9. Symmetry of the stress tensor

Let $\boldsymbol{\sigma}_{ij} = -\mathbf{T}_{ij}$ denote the Cartesian components of the stress tensor in a continuous medium. Consider an infinitesimal cube of medium, whose edges (length $\mathrm{d}\ell$) are parallel to the axes of the coordinate system.

i. Explain why the k-th component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the continuous medium obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\mathbf{T}_{ij}(\mathrm{d}\ell)^3$, with ϵ_{ijk} the usual Levi-Civita symbol.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the medium mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?

[1] For the sake of brevity the dependence of V^r, V^θ, V^z and the basis vectors $\vec{u}_r, \vec{u}_\theta$ on the position $\vec{r}$ is not denoted.

Tutorial sheet 4

The exercise marked with a star is homework.

Discussion topic: What is a perfect fluid? a Newtonian fluid? What are the basic equations governing their respective motions?

*10. General momentum and energy equations in a fluid

In this exercise, the roman indices $(i,j\dots)$ denote the coordinates of vectors and tensors in a Cartesian coordinate system, and ∂_i denotes the partial derivative with respect to the i-th coordinate of the position vector $\vec{r}$ in that system. For brevity, the variables $(t,\vec{r})$ of the fields are omitted, and Einstein's summation convention over repeated indices is used.

i. Momentum equation

Check that the Euler and Navier–Stokes equations are special cases of the more general equation

$$\rho\frac{\mathrm{D}\mathsf{v}^j}{\mathrm{D}t} = \partial_i\boldsymbol{\sigma}^{ij} + f_V^j, \tag{1}$$

with $\boldsymbol{\sigma}^{ij}$ the components of Cauchy's stress tensor and f_V^i those of the external volume forces acting on the fluid. If needed, you can find the so-called "constitutive equations" defining $\boldsymbol{\sigma}^{ij}$ for a perfect and a Newtonian fluid in the lecture notes.

ii. Energy equations

a) Check the identity

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\vec{\mathsf{v}}^2+\frac{e}{\rho}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{\mathsf{v}}^2+e\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\vec{\mathsf{v}}^2+e\right)\vec{\mathsf{v}}\right]. \tag{2}$$

Note that e/ρ is the specific internal energy.

Assuming that the volume forces derive from a time-independent potential energy, $\vec{f}_V = -\rho\vec{\nabla}\Phi$, show the identity

$$\frac{\partial(\rho\Phi)}{\partial t} + \vec{\nabla}\cdot\left(\rho\Phi\vec{\mathsf{v}}\right) = -\vec{f}_V\cdot\vec{\mathsf{v}}. \tag{3}$$

What does the term on the right-hand side represent?

b) Using the identities of the previous question, show that the energy equations given for perfect and Newtonian fluids in the lecture are special cases of the more general equation

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\vec{\mathsf{v}}^2+\frac{e}{\rho}\right) = -\vec{\nabla}\cdot\vec{\jmath}_Q + \partial_i\left(\boldsymbol{\sigma}^{ij}\mathsf{v}_j\right) + \vec{f}_V\cdot\vec{\mathsf{v}}, \tag{4}$$

where $\vec{\jmath}_Q$ denotes the heat flux density.

c) Part of equation (4) is purely mechanical: show that the momentum equation (1) leads to

$$\rho\frac{\mathrm{D}(\frac{1}{2}\vec{\mathsf{v}}^2)}{\mathrm{D}t} = \mathsf{v}_j\partial_i\boldsymbol{\sigma}^{ij} + \vec{f}_V\cdot\vec{\mathsf{v}}, \tag{5}$$

Deduce therefrom the equation governing the rate of change of the specific internal energy. Which form does this equation for the material derivative of e/ρ take in the case of a perfect fluid? Can you deduce an equation for $\mathrm{D}e/\mathrm{D}t$ in perfect fluids, involving the *enthalpy density* $w \equiv e + \mathcal{P}$?

The general equation for e/ρ (or e) can then be used, together with thermodynamic identities, to derive an equation governing the rate of change of the entropy — but not here.

11. Harmonic oscillations in a fluid

Consider the idealized case of a fluid with uniform, yet time dependent mass density: $\rho(t)$. The fluid velocity field is assumed to be spherically symmetric and with a harmonic dependence on time: $\vec{\mathsf{v}}(t,\vec{r}) = \mathsf{v}(r)\cos(\omega_0 t)\,\vec{\mathrm{e}}_r$, where $\vec{\mathrm{e}}_r \equiv \vec{r}/|\vec{r}| \equiv \vec{r}/r$ and ω_0 a given angular frequency.

Determine the mass density $\rho(t)$ and the dependence $\mathsf{v}(r)$.

Hint: In spherical coordinates, the divergence of a purely radial vector field $\vec{V}(\vec{r}) = V_r(r)\,\vec{\mathrm{e}}_r$ is

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\big[r^2 V_r(r)\big].$$

Tutorial sheet 5

The exercise marked with a star is homework.

Discussion topics:

- Give the basic equations governing the dynamics of perfect fluids.
- What is the Bernoulli equation? Give some examples of application.

*12. Simplified model of star

In an oversimplified approach, one may model a star as a sphere of fluid—a plasma—with uniform mass density ρ. This fluid is in mechanical equilibrium under the influence of pressure $\mathcal{P}$ and gravity. Throughout this exercise, the rotation of the star is neglected.

i. Determine the gravitational field at a distance r from the center of the star.

ii. Assuming that the pressure only depend on r, write down the equation expressing the mechanical equilibrium of the fluid. Determine the resulting function $\mathcal{P}(r)$. Compute the pressure at the star center as function of the mass M and radius R of the star. Calculate the numerical value of this pressure for $M = 2 \times 10^{30}$ kg (solar mass) and $R = 7 \times 10^{8}$ m (solar radius).

iii. The matter constituting the star is assumed to be an electrically neutral mixture of hydrogen nuclei and electrons. Show that the order of magnitude of the total particle number density of that plasma is $n \approx 2\rho/m_p$, with m_p the proton mass. Estimate the temperature at the center of the sun.

Hint: $m_p = 1.6 \times 10^{-27}$ kg; $k_B = 1.38 \times 10^{-23}$ J $\cdot$ K^{-1}.

13. Rotating fluid in a uniform gravitational potential

Consider a perfect fluid contained in a straight cylindrical vessel which rotates with constant angular velocity $\vec{\Omega} = \Omega\,\vec{e}_3$ about its vertical axis, the whole system being placed in a uniform gravitational field $-g\,\vec{e}_3$. Assuming that the fluid rotates with the same angular velocity and that its motion is incompressible, determine the shape of the free surface of the fluid.

Hint: Despite the geometry, working with Cartesian coordinates is quite straightforward. At the free surface, the fluid pressure is constant (it equals the atmospheric pressure).

14. Stationary vortex

Let $\vec{\omega}(t,\vec{r}) = A\,\delta(x^1)\,\delta(x^2)\,\vec{e}_3$ be the vorticity field in a fluid, with A a real constant and $\{x^i\}$ Cartesian coordinates. Determine the corresponding flow velocity field $\vec{v}(t,\vec{r})$.

Hint: You should invoke symmetry arguments and Stokes' theorem. A useful formal analogy is provided by the Maxwell–Ampère equation of magnetostatics.

Tutorial sheet 6

The exercises marked with a star are homework.

Discussion topics:

- What is Kelvin's circulation theorem? What does it imply for the vorticity?
- What is a potential flow? What are the corresponding equations of motion?

*15. Statics of rotating fluids

This exercise is strongly inspired by Chapter 13.3.3 of *Modern Classical Physics* by Roger D. Blandford and Kip S. Thorne.

Consider a fluid, bound by gravity, which is rotating rigidly, i.e. with a uniform angular velocity $\vec{\Omega}_0$ with respect to an inertial frame, around a given axis. In a reference frame that co-rotates with the fluid, the latter is at rest, and thus governed by the laws of hydrostatics—except that you now have to consider an additional term...

i. Relying on your knowledge from point mechanics, show that the usual equation of hydrostatics (in an inertial frame) is replaced in the co-rotating frame by

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\big[\Phi(\vec{r}) + \Phi_{\text{cen.}}(\vec{r})\big], \tag{1}$$

where $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big[\vec{\Omega}_0 \times \vec{r}\big]^2$ denotes the potential energy from which derives the centrifugal inertial force density, $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$, while $\Phi(\vec{r})$ is the gravitational potential energy.

ii. Show that Eq. (1) implies that the equipotential lines of $\Phi + \Phi_{\text{cen.}}$ coincide with the contours of constant mass density as well as with the isobars.

iii. Consider a slowly spinning fluid planet of mass M, assuming for the sake of simplicity that the mass is concentrated at the planet center, so that the gravitational potential is unaffected by the rotation. Let R_e resp. R_p denote the equatorial resp. polar radius of the planet, where $|R_e - R_p| \ll R_e \simeq R_p$, and g be the gravitational acceleration at the surface of the planet.
Using questions **i.** and **ii.**, show that the difference between the equatorial and polar radii is

$$R_e - R_p \simeq \frac{R_e^2 |\vec{\Omega}_0|^2}{2g}.$$

Compute this difference in the case of Earth ($R_e \simeq 6.4 \times 10^3$ km)—which as everyone knows behaves as a (viscous) fluid if you look at it long enough—and compare with the actual value.

*16. Model of a tornado

In a simplified approach, one may model a tornado as the steady incompressible flow of a perfect fluid—air—with mass density $\rho = 1.3\ \text{kg}\cdot\text{m}^{-3}$, with a vorticity $\vec{\omega}(\vec{r}) = \omega(\vec{r})\,\vec{e}_3$ which remains uniform inside a cylinder—the "eye" of the tornado—with (vertical) axis along $\vec{e}_3$ and a finite radius $a = 50$ m, and vanishes outside.

i. Express the velocity $\mathsf{v}(r) \equiv \big|\vec{\mathsf{v}}(\vec{r})\big|$ at a distance $r = |\vec{r}|$ from the axis as a function of r and and the velocity $\mathsf{v}_a \equiv \mathsf{v}(r = a)$ at the edge of the eye.
Compute ω inside the eye, assuming $\mathsf{v}_a = 180$ km/h.

ii. Show that for $r > a$ the tornado is equivalent to a vortex at $x^1 = x^2 = 0$ (as in exercise **14**). What is the circulation around a closed curve circling this equivalent vortex?

iii. Assuming that the pressure $\mathcal{P}$ far from the tornado equals the "normal" atmospheric pressure $\mathcal{P}_0$, determine $\mathcal{P}(r)$ for $r > a$. Compute the barometric depression $\Delta\mathcal{P} \equiv \mathcal{P}_0 - \mathcal{P}$ at the edge of the eye. Consider a horizontal roof made of a material with mass surface density 100 kg/m^2: is it endangered by the tornado?

17. Heat diffusion

In a Newtonian fluid at rest, the energy balance equation becomes

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla} \cdot \left[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\right]$$

with e the internal energy density, κ the heat capacity and T the temperature.

Assuming that $C \equiv \partial e/\partial T$ and κ are constant coefficients and introducing $\chi \equiv \kappa/C$, determine the temperature profile $T(t,\vec{r})$ for $z < 0$ with the boundary condition of a uniform, time-dependent temperature $T(t, z = 0) = T_0 \cos(\omega t)$ in the plane $z = 0$. At which depth is the amplitude of the temperature oscillations 10 % of that in the plane $z = 0$?

Tutorial sheet 7

The exercises marked with a star are homework.

Discussion topics:

- What are the fundamental equations governing the dynamics of non-relativistic Newtonian fluids?
- Dynamical similarity and the Reynolds number. You could also educate yourself on the topic of Life at low Reynolds number and the "scallop theorem" by reading E. M. Purcell's article (also accessible via the web page of the lectures)

*18. Taylor–Couette flow. Measurement of shear viscosity

A Couette viscometer consists of an annular gap, filled with fluid, between two concentric cylinders with height L. The outer cylinder (radius R_2) rotates around the common axis with angular velocity Ω_2, while the inner cylinder (radius R_1) remains motionless. The motion of the fluid is assumed to be two-dimensional, laminar, incompressible, and steady.

Throughout this exercise, we use a system of cylinder coordinates (r, φ, z) with the physicists' usual convention, i.e. the corresponding basis vectors are are normalized to unity.

i. Check that the continuity equation leads to $\mathsf{v}^r = 0$, with v^r the radial component of the flow velocity.

ii. Prove that the Navier–Stokes equation lead to the equations

$$\frac{\mathsf{v}^\varphi(r)^2}{r} = \frac{1}{\rho}\frac{\partial \mathcal{P}(r)}{\partial r} \tag{1}$$

$$\frac{\partial^2 \mathsf{v}^\varphi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \mathsf{v}^\varphi(r)}{\partial r} - \frac{\mathsf{v}^\varphi(r)}{r^2} = 0. \tag{2}$$

What is the meaning of Eq. (1)? Solve Eq. (2) with the ansatz $\mathsf{v}^\varphi(r) = ar + \frac{b}{r}$.

iii. One can show that the $r\varphi$-component of the stress tensor is given by

$$\sigma^{r\varphi} = \eta\left(\frac{1}{r}\frac{\partial \mathsf{v}^r}{\partial \varphi} + \frac{\partial \mathsf{v}^\varphi}{\partial r} - \frac{\mathsf{v}^\varphi}{r}\right).$$

Show that $\sigma^{r\varphi} = -\frac{2b\eta}{r^2}$, where b is the same coefficient as above.

iv. A torque $\mathcal{M}_z$ is measured at the surface of the inner cylinder. How can the shear viscosity η of the fluid be deduced from this measurement?
Numerical example: $R_1 = 10$ cm, $R_2 = 11$ cm, $L = 10$ cm, $\Omega_2 = 10\ \mathrm{rad\cdot s^{-1}}$ and $\mathcal{M}_z = 7,246\cdot 10^{-3}$ N·m.

*19. Flow of a Newtonian fluid down a constant slope

A layer of Newtonian fluid is flowing under the influence of gravity (acceleration g) down a slope inclined at an angle α from the horizontal. The fluid itself is assumed to have a constant thickness h, so that its free surface is a plane parallel to its bottom, and the flow is steady, laminar and incompressible. One further assumes that the pressure at the free surface of the fluid as well as "at the ends" at large $|x|$ is constant—i.e., the flow is entirely caused by gravity, not by a pressure gradient.
To fix notations, let x denote the direction along which the fluid flows, with the basis vector oriented downstream, and y be the direction perpendicular to x, oriented upwards.

i. Show that the flow velocity magnitude v and pressure $\mathcal{P}$ of the fluid obey the equations

$$\frac{\partial \mathsf{v}}{\partial x} = 0 \quad , \quad \eta\triangle \mathsf{v} = -\rho g \sin\alpha \quad , \quad \frac{\partial \mathcal{P}}{\partial y} = -\rho g\cos\alpha, \tag{3}$$

with the boundary conditions$\mathsf{v} = 0$ at $y = 0$, $\partial\mathsf{v}/\partial\mathsf{y} = 0$ at $y = h$, and $\mathcal{P} = \mathcal{P}_0$ at $y = h$. Determine the pressure and then the velocity profile.

ii. Compute the rate of volume flow ("volumetric flux") across a surface $\mathcal{S}$ perpendicular to the x-direction.

20. Flow due to an oscillating plane boundary

Consider a rigid infinitely extended plane boundary ($y = 0$) that oscillates in its own plane with a sinusoidal velocity $U\cos(\omega t)\,\vec{e}_x$. The region $y > 0$ is filled with an incompressible Newtonian fluid with uniform kinematic shear viscosity $\nu \equiv \eta/\rho$. We shall assume that volume forces on the fluid are negligible, that the pressure is uniform and remains constant in time, and that the fluid motion induced by the plane oscillations does not depend on the coordinates x, z.

i. Determine the flow velocity $\vec{\mathsf{v}}(t, y)$ and plot the resulting profile.

ii. What is the characteristic thickness of the fluid layer in the vicinity of the plane boundary that follows the oscillations? Comment on your result.

21. Dimensional consideration for viscous flows in a tube

Consider the motion of a given fluid in a cylindrical tube of length L and of circular cross section under the action of a difference $\Delta\mathcal{P}$ between the pressures at the two ends of the tube. The relation between the pressure drop per unit length $\Delta\mathcal{P}/L$ and the magnitude of the mean velocity $\langle\mathsf{v}\rangle$—defined as the average over a cross section of the tube—is given by

$$\frac{\Delta\mathcal{P}}{L} = C\langle\mathsf{v}\rangle^n,$$

with C a constant that depends on the fluid mass density ρ, on the kinematic shear viscosity ν, and on the radius a of the tube cross section. n is a number which depends on the type of flow: $n = 1$ if the flow is laminar (this is the Hagen–Poiseuille law seen in the lecture), while measurements in turbulent flows by Hagen (1854) resp. Reynolds (1883) have given $n = 1.75$ resp. $n = 1.722$.

Assuming that C is—up to a pure number—a product of powers of ρ, ν and a, determine the exponents of these power laws using dimensional arguments.

Tutorial sheet 8

The exercise marked with a star is homework.

Discussion topics: What is a sound wave? How do you derive the corresponding equation of motion? How is the speed of sound defined? What happens when the wave amplitude becomes large? What is the effect of viscous effects?

*22. One-dimensional "similarity flow"

Consider a perfect fluid at rest in the region $x \geq 0$ with pressure $\mathcal{P}_0$ and mass density ρ_0; the region $x < 0$ is empty ($\mathcal{P} = 0$, $\rho = 0$). At time $t = 0$, the wall separating both regions is removed, so that the fluid starts flowing into the region $x < 0$. The goal of this exercise is to solve this instance of *Riemann's problem* by determining the flow velocity $\mathsf{v}(t, x)$ for $t > 0$. It will be assumed that the pressure and mass density of the fluid remain related by

$$\frac{\mathcal{P}}{\mathcal{P}_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \qquad \text{with } \gamma > 1$$

throughout the motion. This relation also gives you the speed of sound $c_s(\rho)$.

i. Assume that the dependence on t and x of the various fields involves only the combination $u \equiv x/t$.[1] Show that the continuity and Euler equations can be recast as

$$\begin{aligned} \left[u - \mathsf{v}(u)\right] \rho'(u) &= \rho(u)\, \mathsf{v}'(u) \\ \rho(u)\left[u - \mathsf{v}(u)\right] \mathsf{v}'(u) &= c_s^2(\rho(u))\, \rho'(u), \end{aligned}$$

where ρ' resp. v' denote the derivative of ρ resp. v with respect to u.

ii. Show that the velocity is either constant, or obeys the equation $u - \mathsf{v}(u) = c_s(\rho(u))$, in which case the squared speed of sound takes the form $c_s^2(\rho) = c_s^2(\rho_0)(\rho/\rho_0)^{\gamma-1}$.

iii. Show that the results of **i.** and **ii.** lead to the relation

$$\mathsf{v}(u) = a + \frac{2}{\gamma - 1}\, c_s(\rho(u)),$$

where a denotes a constant whose value is fixed by the condition that $\mathsf{v}(u)$ remain continuous inside the fluid. Show eventually that in some interval for the values of u, the norm of v is given by

$$|\mathsf{v}(u)| = \frac{2}{\gamma + 1}\left[c_s(\rho_0) - u\right].$$

iv. Sketch the profiles of the mass density $\rho(u)$ and the streamlines $x(t)$ and show that after the removal of the separation at $x = 0$ the information propagates with velocity $2c_s(\rho_0)/(\gamma - 1)$ towards the negative-x region, while it moves to the right with the speed of sound $c_s(\rho)$.

23. Inviscid Burgers equation

The purpose of this exercise is to show how an innocent-looking—yet non-linear—partial differential equation with a smooth initial condition may lead after finite amount of time to a discontinuity, i.e. a shock wave.

Neglecting the pressure term in the one-dimensional Euler equation leads to the so-called *inviscid Burgers equation*

$$\frac{\partial \mathsf{v}(t, x)}{\partial t} + \mathsf{v}(t, x)\frac{\partial \mathsf{v}(t, x)}{\partial x} = 0.$$

[1]... which is what is meant by "self-similar".

i. Show that the solution with (arbitrary) given initial condition $\mathsf{v}(0,x)$ for $x \in \mathbb{R}$ obeys the implicit equation $\mathsf{v}(0,x) = \mathsf{v}\big(t, x + \mathsf{v}(0,x)\,t\big)$.

Hint: `http://en.wikipedia.org/wiki/Burgers'_equation`

ii. Consider the initial condition $\mathsf{v}(0,x) = \mathsf{v}_0\,\mathrm{e}^{-(x/x_0)^2}$ with v_0 and x_0 two real numbers. Show that the flow velocity becomes discontinuous at time $t = \sqrt{\mathrm{e}/2}\,x_0/\mathsf{v}_0$, namely at $x = x_0\sqrt{2}$.

Tutorial sheet 9

The exercise marked with a star is homework.

*24. Instabilities in parallel shear flows

In the lectures we considered a number of simple steady incompressible flows with velocity of the form $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(y)\vec{\mathrm{e}}_x$, where x, y, z are Cartesian coordinates. For the stability of such so-called "parallel shear flows" there exist a number of results, some of which are discussed in this exercise. Throughout we assume that the mass density ρ_0 remains uniform and constant, and that there are no external forces.

i. Starting from the continuity and incompressible Navier–Stokes equations, write down the linearized equations of motion governing the evolution of perturbations $\delta\vec{\mathsf{v}}(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ of steady fields $\vec{\mathsf{v}}_0(\vec{r})$ and $\mathcal{P}_0(\vec{r})$, assuming $\vec{\mathsf{v}}_0(\vec{r}) = \mathsf{v}_0(y)\vec{\mathrm{e}}_x$.

One can show (Squire's theorem) that it is sufficient to investigate perturbations that are two-dimensional, i.e. that do not depend on z and such that $\delta\vec{\mathsf{v}}$ lies in the (x, y)-plane. To describe the latter, one can introduce the associated stream function $\psi(t,\vec{r})$, such that the non-zero components of $\delta\vec{\mathsf{v}}$ are given by $\delta\mathsf{v}^x = -\partial\psi/\partial y$ and $\delta\mathsf{v}^y = \partial\psi/\partial x$.

ii. Assume first that the fluid is perfect.

a) Using the linearized equations of motion you obtained in **i.**, show that the stream function satisfies the partial differential equation

$$\left[\frac{\partial}{\partial t} + \mathsf{v}_0(y)\frac{\partial}{\partial x}\right]\triangle\psi(t,\vec{r}) - \frac{\partial^2\mathsf{v}_0(y)}{\partial y^2}\frac{\partial\psi(t,\vec{r})}{\partial x} = 0. \tag{1}$$

b) Making the Fourier ansatz $\psi(t,\vec{r}) = \widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}$, show that Eq. (1) leads to *Rayleigh's equation*

$$\big[\mathsf{v}_0(y) - c(k)\big]\left(\frac{\partial^2}{\partial y^2} - k^2\right)\widetilde{\psi}(y) - \frac{\partial^2\mathsf{v}_0(y)}{\partial y^2}\widetilde{\psi}(y) = 0, \tag{2}$$

where $c(k) \equiv \omega/k$.

For a given profile $\mathsf{v}_0(y)$ of the unperturbed flow and a fixed wavenumber k, this is an eigenvalue equation, whose solutions are eigenfunctions $\widetilde{\psi}(y)$ with associated eigenvalues $c(k)$. Show that if $\widetilde{\psi}$ is an eigenfunction associated with some eigenvalue $c(k)$, then its complex conjugate $\widetilde{\psi}^*$ is also an eigenfunction, with eigenvalue $c(k)^*$. What does this mean for the stability of the unperturbed flow in case one of the eigenvalues is not real?

iii. If you still have time, you may show that in a Newtonian incompressible fluid, Rayleigh's equation is replaced by the *Orr–Sommerfeld equation*

$$\big[\mathsf{v}_0(y) - c(k)\big]\left(\frac{\partial^2}{\partial y^2} - k^2\right)\widetilde{\psi}(y) - \frac{\partial^2\mathsf{v}_0(y)}{\partial y^2}\widetilde{\psi}(y) = \frac{\nu}{\mathrm{i}k}\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2\widetilde{\psi}(y), \tag{3}$$

with $\nu \equiv \eta/\rho_0$ the kinematic shear viscosity of the fluid.

25. Instability of the viscous Burgers equation

Neglecting the pressure term in the Navier–Stokes equation for a one-dimensional incompressible problem without external force yields the so-called *viscous Burgers equation*[1]

$$\frac{\partial\mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\frac{\partial\mathsf{v}(t,x)}{\partial x} = \nu\frac{\partial^2\mathsf{v}(t,x)}{\partial x^2}, \tag{4}$$

[1] You already encountered its inviscid version in exercise **23.**

where $\nu \equiv \eta/\rho$ is the kinematic shear viscosity of the fluid. A trivial solution to this equation of motion is the steady uniform flow $\mathsf{v}(t,x) = \mathsf{v}_0$.

Let us add a perturbation $\delta\mathsf{v}(t,x)$.

a) Write down the linearized equation of motion governing the evolution of $\delta\mathsf{v}$ and derive the corresponding dispersion relation using an appropriate Fourier ansatz.

b) Fixing first $k \in \mathbb{R}$, check that the perturbation is exponentially damped in time.

c) Consider now a fixed $\omega \in \mathbb{R}$. How does the perturbation propagate along the x-direction? (*Hint*: For the sake of simplicity you may restrict your discussion to the small-viscosity case $\omega\nu \ll \mathsf{v}_0^2$.)

Tutorial sheet 10

The exercise marked with a star is homework.

Discussion topic: Turbulence in fluids: what is it? Why does it require a Reynolds number larger than some critical value to develop? In fully developed turbulence, what are the mean flow, the fluctuating flow, the Reynolds stress tensor, the energy cascade?

*26. A mathematical model to reproduce some features of fully developed turbulence

While trying to solve the problem of incompressible turbulence in fluids, Burgers (again!) wrote down a system of simpler equations—a toy mathematical model—that share a few features of the dynamical equations governing the mean flow and the flow fluctuations, namely

$$\frac{\mathrm{d}\bar{\mathsf{v}}(t)}{\mathrm{d}t} = \mathcal{P} - \mathsf{v}'(t)^2 - \nu\,\bar{\mathsf{v}}(t), \tag{1a}$$

$$\frac{\mathrm{d}\mathsf{v}'(t)}{\mathrm{d}t} = \bar{\mathsf{v}}(t)\mathsf{v}'(t) - \nu\,\mathsf{v}'(t), \tag{1b}$$

with $\bar{\mathsf{v}}$, v' two unknown functions, while ν is a parameter and $\mathcal{P}$ a constant. In these equations, all quantities (including t) are dimensionless and real.
The questions **i.**, **ii.**, **iii.**, **iv.** are to a very large extent independent from each other.

i. Enumerate the similarities between Burgers' set of equations and the "true" ones given in the lecture. That is, identify the physical content of each term in Eqs. (1), and recognize how key mathematical features of the fluid dynamical equations are seemingly reproduced—while others are obviously not, which may deserve a discussion as well.

ii. Viewing $\bar{\mathsf{v}}$, v' as velocities, write down the differential equation governing the evolution of the sum of the associated kinetic energies (per unit mass...). Note that the terms which you obtain have a straightforward physical interpretation, which smoothly matches those found in question **i.**

iii. "Laminar" solution

a) Show that equations (1) admit a set of stationary solutions with a finite "mean flow velocity" $\bar{\mathsf{v}} = \bar{\mathsf{v}}_0$ and a vanishing "fluctuating velocity" v'.

b) Check that these solutions are stable as long as $\mathcal{P} < \nu^2$. That is, any perturbation $(\delta\bar{\mathsf{v}}, \delta\mathsf{v}')$ yielding total velocities $\bar{\mathsf{v}}(t) = \bar{\mathsf{v}}_0 + \delta\bar{\mathsf{v}}(t)$, $\mathsf{v}'(t) = \delta\mathsf{v}'(t)$ will be exponentially damped. On the other hand, the solution $(\bar{\mathsf{v}} = \bar{\mathsf{v}}_0,\ \mathsf{v}' = 0)$ is unstable for $\mathcal{P} > \nu^2$.

iv. "Turbulent" solution
Let us now assume $\mathcal{P} > \nu^2$.

a) Show that equations (1) now admit two sets of stationary solutions, both involving a finite mean flow velocity $\bar{\mathsf{v}}$—the same for both sets—and a finite fluctuating velocity $\mathsf{v}' = \pm\mathsf{v}'_0$.

b) Show that both solutions are stable for $\mathcal{P} > \nu^2$.
Hint: You should have to distinguish two cases, namely $\nu < \mathcal{P} \leq \frac{9}{8}\nu^2$ and $\mathcal{P} > \frac{9}{8}\nu^2$.

The appearance of several regimes—one laminar ($\mathsf{v}' = 0$), the other turbulent ($\mathsf{v}' \neq 0$)—depending on the value of a parameter is reminiscent of the onset of turbulence above a geometry-dependent given Reynolds number in the real fluid dynamical case: in that respect, Burgers' toy model reproduces an important feature of the true equations. On the other hand, the existence of two competing turbulent solutions above the critical parameter value is an over-simplification of the real turbulent motion.

27. Dynamics of the mean flow in fully developed turbulence

The velocity field resp. pressure for an incompressible turbulent flow is split into an average and a fluctuating part as

$$\vec{\mathsf{v}}(t,\vec{r}) = \overline{\vec{\mathsf{v}}}(t,\vec{r}) + \vec{\mathsf{v}}'(t,\vec{r}) \quad \text{resp.} \quad \mathcal{P}(t,\vec{r}) = \overline{\mathcal{P}}(t,\vec{r}) + \mathcal{P}'(t,\vec{r}),$$

where the motion with $\overline{\vec{\mathsf{v}}}$, $\overline{\mathcal{P}}$ is referred to as "mean flow". For the sake of simplicity, a system of Cartesian coordinates is being assumed—the components of the gradient thus involve partial derivatives, instead of the more general covariant derivatives. Throughout the exercise, Einstein's summation convention over repeated indices is used.

Check that the incompressible Navier–Stokes equation obeyed by $\vec{\mathsf{v}}$ and $\mathcal{P}$ leads for the mean-flow quantities to the equation

$$\frac{\partial \overline{\mathsf{v}^i}}{\partial t} + \big(\overline{\vec{\mathsf{v}}} \cdot \vec{\nabla}\big)\overline{\mathsf{v}^i} = -\frac{1}{\rho}\frac{\partial \overline{\mathcal{P}}}{\partial x_i} - \frac{\partial \overline{\mathsf{v}'^i \mathsf{v}'^j}}{\partial x^j} + \nu \triangle \overline{\mathsf{v}^i}. \tag{2}$$

Show that this gives for the kinetic energy per unit mass $\overline{k} \equiv \frac{1}{2}\overline{\vec{\mathsf{v}}}^2$ associated with the mean flow the evolution equation

$$\frac{\partial \overline{k}}{\partial t} + \big(\overline{\vec{\mathsf{v}}} \cdot \vec{\nabla}\big)\overline{k} = -\frac{\partial}{\partial x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\overline{\mathsf{v}^j} + \Big(\overline{\mathsf{v}'^i \mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathsf{v}_i}\right] + \Big(\overline{\mathsf{v}'^i \mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathbf{S}_{ij}} \tag{3}$$

with $\overline{\mathbf{S}^{ij}} \equiv \frac{1}{2}\left(\frac{\partial \overline{\mathsf{v}^i}}{\partial x_j} + \frac{\partial \overline{\mathsf{v}^j}}{\partial x_i} - \frac{2}{3}g^{ij}\vec{\nabla}\cdot\overline{\vec{\mathsf{v}}}\right)$ the components of the (mean) rate-of-shear tensor.

Tutorial sheet 11

The exercise marked with a star is homework.

Discussion topics:

– Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects play a role?
– What are the fundamental equations of the dynamics of a relativistic fluid? What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?

*28. (1+1)-dimensional relativistic motion

Consider a (1+1)-dimensional relativistic motion along a direction denoted as z, where the denomination "1+1" stands for one time and one spatial dimension. Throughout the exercise, the other two spatial directions play no role and the corresponding variables x, y are totally omitted. In addition, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein's summation convention over repeated indices.

To describe the physics, one may naturally use Minkowski coordinates $(x^0, x^3) = (t, z)$, with corresponding derivatives $(\partial_0, \partial_3) = (\partial/\partial t, \partial/\partial z)$. If there is a high-velocity motion in the z-direction, a better choice might be to use the *proper time* τ and *spatial rapidity* ς such that[1]

$$x^{0'} \equiv \tau \equiv \sqrt{t^2 - z^2}, \quad x^{3'} \equiv \varsigma \equiv \frac{1}{2}\log\frac{t+z}{t-z} \quad \text{where } |z| \leq t. \tag{1}$$

The partial derivatives with respect to these new coordinates will be denoted $(\partial_{0'}, \partial_{3'}) = (\partial/\partial\tau, \partial/\partial\varsigma)$.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau\cosh\varsigma, \quad z = \tau\sinh\varsigma. \tag{2}$$

(*Hint:* Recognize $\frac{1}{2}\log\frac{1+u}{1-u}$).

ii. In a change of coordinates $\{x^\mu\} \to \{x^{\mu'}\}$, the contravariant components V^μ of a 4-vector transform according to $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$ (with summation over ν!) where $\Lambda^{\mu'}{}_\nu \equiv \partial x^{\mu'}/\partial x^\nu$.

Compute first from Eq. (2) the matrix elements $\Lambda^\nu{}_{\mu'} \equiv \partial x^\nu/\partial x^{\mu'}$ (with $\nu \in \{0,3\}$, $\mu' \in \{0', 3'\}$) of the inverse transformation $\{V^{\mu'}\} \to \{V^\mu\}$. Inverting the 2×2-matrix you thus found, deduce the following relationship between the components of the 4-vector in the two coordinate systems

$$\begin{cases} V^{0'} = \cosh\varsigma\, V^0 - \sinh\varsigma\, V^3 \\ V^{3'} = -\dfrac{1}{\tau}\sinh\varsigma\, V^0 + \dfrac{1}{\tau}\cosh\varsigma\, V^3. \end{cases} \tag{3}$$

iii. Using the relation $\partial_\nu = \Lambda^{\mu'}{}_\nu \partial_{\mu'}$ and the matrix elements $\{\Lambda^{\mu'}{}_\nu\}$ you found in **ii.**—and which can be read off Eq. (3)—, express the "4-divergence" $\partial_\nu V^\nu$ of a 4-vector field V^ν in terms of the partial derivatives $\partial_{\mu'}$ and the components $V^{\mu'}$ in the (τ, ς)-system.

You should find a result that does not equal $\partial_{\mu'}V^{\mu'} = \partial_\tau V^\tau + \partial_\varsigma V^\varsigma$, which is why in the lecture notes the notation $\mathrm{d}_{\mu'}V^{\mu'}$ is used for the 4-divergence in an arbitrary coordinate system.

iv. Draw on a spacetime diagram—with t on the vertical axis and z on the horizontal axis—the lines of constant τ and those of constant ς.

Remark: The coordinates (τ, ς) are sometimes called *Milne coordinates*.

[1] ς = `\varsigma` is the word-final form for the lower case sigma, not to be confused with ζ (zeta).

29. Quantum number conservation

Consider a 4-current with components $N^\mu(\mathsf{x})$ obeying the continuity equation $\partial_\mu N^\mu(\mathsf{x}) = 0$, where the $\{x^\mu\}$ hidden in the notation ∂_μ are Minkowski coordinates. Show that the quantity

$$\mathcal{N} = \frac{1}{c}\int N^0(\mathsf{x})\,\mathrm{d}^3\vec{r}$$

is a Lorentz scalar, by convincing yourself first that it can be rewritten in the form

$$\mathcal{N} = \frac{1}{c}\int_{x^0=\text{const.}} N^\mu(\mathsf{x})\,\mathrm{d}^3\sigma_\mu, \tag{4}$$

where $\mathrm{d}^3\sigma_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ is a 4-vector, with $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ the antisymmetric 4-tensor defined by

$$\mathrm{d}^3\mathcal{V}^{012} = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2, \quad \mathrm{d}^3\mathcal{V}^{021} = -\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^1, \quad \text{etc.}$$

and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric Levi–Civita symbol with the convention $\epsilon_{0123} = +1$, such that $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ represents the 3-dimensional hypersurface element in Minkowski space.

Tutorial sheet 12

The exercises marked with a star are homework.

Discussion topics:

– Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects play a role?
– What are the fundamental equations of the dynamics of a relativistic fluid? What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?

*30. Energy-momentum tensor

Let $\mathcal{R}$ denote a fixed reference frame. Consider a perfect fluid whose local rest frame at a point x moves with velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}$. Show with the help of a Lorentz transformation that the Minkowski components of the energy-momentum tensor of the fluid at x are given to order $\mathcal{O}(|\vec{\mathsf{v}}|/c)$ by

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & \mathcal{P} & 0 & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & 0 & \mathcal{P} & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} & 0 & 0 & \mathcal{P} \end{pmatrix},$$

where for the sake of brevity the x-dependence of the various fields is omitted. Check the compatibility of this result with the general formula for $T^{\mu\nu}$ given in the lecture.

*31. A family of solutions of the dynamical equations for perfect relativistic fluids

Let $\{x^\mu\}$ denote Minkowski coordinates and $\tau^2 \equiv -x^\mu x_\mu$, where the "mostly plus" metric is used. Show that the following four-velocity, pressure and charge density constitute a solution of the equations describing the motion of a perfect relativistic fluid with equation of state $\mathcal{P} = K\varepsilon$ and a single conserved charge:

$$u^\mu(\mathsf{x}) = \frac{x^\mu}{\tau} \quad , \quad \mathcal{P}(\mathsf{x}) = \mathcal{P}_0\left(\frac{\tau_0}{\tau}\right)^{3(1+K)} \quad , \quad n(\mathsf{x}) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}\big(\sigma(\mathsf{x})\big), \tag{1}$$

with τ_0, $\mathcal{P}_0$, n_0 arbitrary constants and $\mathcal{N}$ an arbitrary function of a single argument, while σ is a function of spacetime coordinates with vanishing comoving derivative: $u^\mu\partial_\mu\sigma(\mathsf{x}) = 0$.

32. Equations of motion of a perfect relativistic fluid

In this exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

Hint: If the covariant derivatives d_μ upset you, assume you have chosen Minkowski coordinates, in which $\mathrm{d}_\mu = \partial_\mu$.

i. Check that the tensor with components $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ defines a projector on the subspace orthogonal to the 4-velocity.
Denoting by d_μ the components of the (covariant) 4-gradient, we define $\nabla^\nu \equiv \Delta^{\mu\nu}\mathrm{d}_\mu$. Can you see the rationale behind this notation?

ii. Show that the energy-momentum conservation equation for a perfect fluid is equivalent to the two equations

$$u^\mu \mathrm{d}_\mu\epsilon + (\epsilon+\mathcal{P})\mathrm{d}_\mu u^\mu = 0 \quad \text{and} \quad (\epsilon+\mathcal{P})u^\mu\mathrm{d}_\mu u^\nu + \nabla^\nu\mathcal{P} = 0. \tag{2}$$

Which known equation does the second one evoke?

33. Vorticity in a perfect relativistic fluid

Consider the *kinematic vorticity tensor* defined by its components

$$\omega_{\mu\nu} \equiv \frac{1}{2}\Delta^{\alpha}_{\ \mu}\Delta^{\beta}_{\ \nu}\big(\mathrm{d}_\beta u_\alpha - \mathrm{d}_\alpha u_\beta\big), \tag{3}$$

where $\Delta^{\alpha}_{\ \mu} \equiv g^{\alpha}_{\ \mu} + u^\alpha u_\mu$ (see exercise **32.i.**).

a) Why is no calculation necessary to prove the identity $\Delta^{\mu\nu}\omega_{\mu\nu} = 0$? Show the identity

$$\omega_{\mu\nu} = \frac{1}{2}\big(\mathrm{d}_\nu u_\mu - \mathrm{d}_\mu u_\nu + a_\mu u_\nu - a_\nu u_\mu\big), \tag{4}$$

where $a^\mu \equiv u^\nu \mathrm{d}_\nu u^\mu$.

b) Define a four-vector by[1] $\omega^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\omega_{\rho\sigma}u_\nu$. What are its components in the local rest frame? What do you recognize?

c) Show that if $a^\mu = 0$, then the vorticity 4-vector obeys the evolution equation

$$u^\nu \mathrm{d}_\nu \omega^\mu = -\frac{2}{3}(\mathrm{d}_\rho u^\rho)\omega^\mu + S^{\mu}_{\ \nu}\omega^\nu, \tag{5}$$

where $S^{\mu\nu}$ is the rate-of-shear tensor.

[1] The convention $\epsilon^{0123} = -1$ is used.

Tutorial sheet 13

The exercise marked with a star is homework.

*34. Speed of sound in ultrarelativistic matter

Consider a perfect fluid with the usual energy-momentum tensor. $T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (\epsilon + \mathcal{P})u^\mu u^\nu/c^2$. It is assumed that there is no conserved quantum number relevant for thermodynamics, so that the energy density in the local rest frame ϵ is function of a single thermodynamic variable, for instance $\epsilon = \epsilon(\mathcal{P})$. Throughout the exercise, Minkowski coordinates are used.

A background "flow" with uniform local-rest-frame energy density and pressure ϵ_0 and $\mathcal{P}_0$ is submitted to a small perturbation resulting in $\epsilon = \epsilon_0 + \delta\epsilon$, $\mathcal{P} = \mathcal{P}_0 + \delta\mathcal{P}$, and $\vec{\mathrm{v}} = \vec{0} + \delta\vec{\mathrm{v}}$.

i. Starting from the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, show that linearization to first order in the perturbations leads to the two equations of motion $\partial_t\delta\epsilon = -(\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\delta\vec{\mathrm{v}}$ and $(\epsilon_0 + \mathcal{P}_0)\partial_t\delta\vec{\mathrm{v}} = -c^2\vec{\nabla}\delta\mathcal{P}$.

ii. Show that the speed of sound is given by the expression $c_s^2 = \dfrac{c^2}{\mathrm{d}\epsilon/\mathrm{d}\mathcal{P}}$.

iii. Compute c_s for a fluid obeying the Stefan–Boltzmann law[1] $\mathcal{P} = \dfrac{g\pi^2}{90}\dfrac{(k_\mathrm{B}T)^4}{(\hbar c)^3}$, with g the number of degrees of freedom (e.g. $g = 2$ for blackbody radiation).

Hint: You may find the Gibbs–Duhem relation useful...

[1]This is a good opportunity to refresh your knowledge on the statistical physics of relativistic systems. Can you give a physical argument why quantum effects always play a role in such systems, as signaled by the presence of $\hbar$ in the equation of state?