X.4.5 Second order dissipative relativistic fluid dynamics

To remedy the instability of the usual Landau–Lifshitz or Eckart formulations of first-order dissipative relativistic fluid dynamics—which is especially a problem for numerical implementations, in which rounding errors will quickly propagate if the theory is unstable—, theories going beyond a first-order expansion in gradients were developed.

Coming back to an arbitrary 4-velocity $\mathsf{u}(\mathsf{x})$, the components of the entropy 4-current $\mathsf{S}(\mathsf{x})$ in a first-order dissipative theory read

$$S^\mu(\mathsf{x}) = \frac{\mathcal{P}(\mathsf{x})g^{\mu\nu}(\mathsf{x}) - T^{\mu\nu}(\mathsf{x})}{T(\mathsf{x})}u_\nu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}N_a^\mu(\mathsf{x}), \tag{X.52a}$$

or equivalently

$$S^\mu(\mathsf{x}) = s(\mathsf{x})u^\mu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\nu_a^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}q^\mu(\mathsf{x}) \tag{X.52b}$$

which reduces to the expression between square brackets on the left hand side of Eq. (X.49b) with Landau's choice of 4-velocity.

This entropy 4-current is *linear* in the dissipative 4-currents $\nu(\mathsf{x})$ and $\mathsf{q}(\mathsf{x})$. In addition, it is independent of the velocity 3-gradients—encoded in the expansion rate $\boldsymbol{\nabla}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})$ and the rate-of-shear tensor $\mathbf{S}(\mathsf{x})$—, which play a decisive role in dissipation. That is, the form (X.52) can be generalized. A more general form for the entropy 4-current is thus

$$\mathsf{S}(\mathsf{x}) = s(\mathsf{x})\mathsf{u}(\mathsf{x}) - \frac{\mu_N(\mathsf{x})}{T(\mathsf{x})}\mathsf{v}(\mathsf{x}) + \frac{1}{T(\mathsf{x})}\mathsf{q}(\mathsf{x}) + \frac{1}{T(\mathsf{x})}\mathsf{Q}(\mathsf{x}) \tag{X.53a}$$

or equivalently, component-wise,

$$S^\mu(\mathsf{x}) = s(\mathsf{x})u^\mu(\mathsf{x}) - \frac{\mu_N(\mathsf{x})}{T(\mathsf{x})}\nu^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}q^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}Q^\mu(\mathsf{x}), \tag{X.53b}$$

with $\mathsf{Q}(\mathsf{x})$ a 4-vector, with components $Q^\mu(\mathsf{x})$, that depends on the flow 4-velocity and its gradients—where $\boldsymbol{\nabla}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})$ and $\mathbf{S}(\mathsf{x})$ are traditionally replaced by $\Pi(\mathsf{x})$ and $\boldsymbol{\pi}(\mathsf{x})$—and on the dissipative currents:

$$Q^\mu(\mathsf{x}) = Q^\mu\big(\mathsf{u}(\mathsf{x}), \mathsf{v}(\mathsf{x}), \mathsf{q}(\mathsf{x}), \Pi(\mathsf{x}), \boldsymbol{\pi}(\mathsf{x})\big). \tag{X.53c}$$

In second order dissipative relativistic fluid dynamics with for simplicity a single conserved charge, the most general form for the additional 4-vector $\mathsf{Q}(\mathsf{x})$ contributing to the entropy density is [49, 50, 51]

$$\mathsf{Q}(\mathsf{x}) = \frac{\beta_0(\mathsf{x})\Pi(\mathsf{x})^2 + \beta_1(\mathsf{x})\mathsf{q}_N(\mathsf{x})^2 + \beta_2(\mathsf{x})\boldsymbol{\pi}(\mathsf{x}):\boldsymbol{\pi}(\mathsf{x})}{2T(\mathsf{x})}\mathsf{u}(\mathsf{x}) - \frac{\alpha_0(\mathsf{x})}{T(\mathsf{x})}\Pi(\mathsf{x})\mathsf{q}_N(\mathsf{x}) - \frac{\alpha_1(\mathsf{x})}{T(\mathsf{x})}\boldsymbol{\pi}(\mathsf{x})\cdot\mathsf{q}_N(\mathsf{x}), \tag{X.54a}$$

where

$$\mathsf{q}_N(\mathsf{x}) \equiv \mathsf{q}(\mathsf{x}) - \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{n(\mathsf{x})}\mathsf{v}(\mathsf{x});$$

component-wise, this reads

$$Q^\mu(\mathsf{x}) = \frac{\beta_0(\mathsf{x})\Pi(\mathsf{x})^2 + \beta_1(\mathsf{x})\mathsf{q}_N(\mathsf{x})^2 + \beta_2(\mathsf{x})\pi_{\nu\rho}(\mathsf{x})\pi^{\nu\rho}(\mathsf{x})}{2T(\mathsf{x})}u^\mu(\mathsf{x}) - \frac{\alpha_0(\mathsf{x})}{T(\mathsf{x})}\Pi(\mathsf{x})q_N^\mu(\mathsf{x}) - \frac{\alpha_1(\mathsf{x})}{T(\mathsf{x})}\pi^\mu{}_\rho(\mathsf{x})q_N^\rho(\mathsf{x}). \tag{X.54b}$$

The 4-vector $\mathsf{Q}(\mathsf{x})$ is now quadratic ("of second order") in the dissipative currents—in the wider sense—$\mathsf{q}(\mathsf{x})$, $\mathsf{v}(\mathsf{x})$, $\Pi(\mathsf{x})$ and $\boldsymbol{\pi}(\mathsf{x})$, and involves 5 additional coefficients depending on temperature and particle-number density, α_0, α_1, β_0, β_1, and β_2.

Substituting this form of $\mathsf{Q}(\mathsf{x})$ in the entropy 4-current (X.53), the simplest way to ensure that its 4-divergence should be positive is to postulate *linear* relationships between the dissipative currents and the gradients of velocity, chemical potential (or rather of $-\mu/T$), and temperature (or rather, $1/T$), as was done in Eqs. (X.50). This recipe yields differential equations for $\Pi(\mathsf{x})$, $\boldsymbol{\pi}(\mathsf{x})$, $\mathsf{q}_N(\mathsf{x})$, representing 9 coupled scalar equations of motion. These describe the relaxation—with appropriate characteristic time scales τ_Π, $\tau_{\boldsymbol{\pi}}$, τ_{q_N} respectively proportional to β_0, β_2, β_1, while the involved "time derivative" is that in the local rest frame, $\mathsf{u}\cdot\mathsf{d}$—, of the dissipative currents towards their first-order expressions (X.50).

Adding up the new equations to the usual ones (X.2) and (X.7), the resulting set of equations, known as (Müller[bd]–)Israel[be]–Stewart[bf] theory, is no longer plagued by the issues that affects the relativistic Navier–Stokes–Fourier equations.

Bibliography for Chapter X

- Andersson & Comer [52];
- Landau–Lifshitz [4, 5], Chapter XV, § 133,134 (perfect fluid) and § 136 (dissipative fluid);
- Romatschke [53];
- Weinberg [54], Chapter 2, § 10 (perfect fluid) and § 11 (dissipative fluid).

[bd] I. Müller, born 1936 [be] W. Israel, 1931–2022 [bf] J. M. Stewart, 1943–2016