X.4.3 General equations of motion

By substituting the decompositions (X.36), (X.40) into the generic conservation laws (X.2), (X.7), one can obtain model-independent equations of motion, that do not depend on any assumption on the various dissipative currents.

For that purpose, let us introduce the notation

$$\nabla^{\mu}(\mathsf{x}) \equiv \Delta^{\mu\nu}(\mathsf{x})\mathrm{d}_{\nu}, \tag{X.45a}$$

where d_{ν}, $\nu \in \{0,1,2,3\}$ denotes the components of the 4-gradient d—involving covariant derivatives in case a non-Minkowski system of coordinates is being used. In geometric formulation, this definition reads

$$\boldsymbol{\nabla}(\mathsf{x}) \equiv \boldsymbol{\Delta}(\mathsf{x}) \cdot \mathsf{d}. \tag{X.45b}$$

As is most obvious in the local rest frame at point x, in which the timelike component $\nabla^0(\mathsf{x})$ vanishes, $\boldsymbol{\nabla}(\mathsf{x})$ is the projection of the gradient on the space-like 3-space orthogonal to the 4-velocity. Using this 3-gradient, the 4-gradient can be written $\mathsf{d} = -\mathsf{u}(\mathsf{u} \cdot \mathsf{d}) + \boldsymbol{\nabla}$, i.e. in terms of components

$$\mathrm{d}_{\mu} = -u_{\mu}(\mathsf{u} \cdot \mathsf{d}) + \nabla_{\mu}. \tag{X.45c}$$

Recognizing that $\mathsf{u} \cdot \mathsf{d}$ is the derivative with respect to (proper) time in the local rest frame, this decomposition has a clear meaning.

The charge conservation equation (X.2) first yields

$$\mathrm{d}_\mu N_a^\mu(\mathsf{x}) = u^\mu(\mathsf{x})\mathrm{d}_\mu n_a(\mathsf{x}) + n_a(\mathsf{x})\mathrm{d}_\mu u^\mu(\mathsf{x}) + \mathrm{d}_\mu \nu_a^\mu(\mathsf{x}) = 0. \tag{X.46a}$$

In turn, the conservation (X.7) of the energy momentum tensor (X.37), projected perpendicular to resp. along the 4-velocity, gives

$$\Delta^\rho{}_\nu(\mathsf{x})\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]u^\mu(\mathsf{x})\mathrm{d}_\mu u^\rho(\mathsf{x}) + \nabla^\rho(\mathsf{x})\mathcal{P}(\mathsf{x}) + \Delta^\rho{}_\nu(\mathsf{x})\mathrm{d}_\mu \tau^{\mu\nu}(\mathsf{x}) = 0 \tag{X.46b}$$

resp.

$$u_\nu(\mathsf{x})\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = -u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) - \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) + u_\nu(\mathsf{x})\mathrm{d}_\mu\tau^{\mu\nu}(\mathsf{x}) = 0.$$

In the latter equation, one can substitute the rightmost term by

$$u_\nu(\mathsf{x})\mathrm{d}_\mu\tau^{\mu\nu}(\mathsf{x}) = \mathrm{d}_\mu\big[u_\nu(\mathsf{x})\tau^{\mu\nu}(\mathsf{x})\big] - \tau^{\mu\nu}(\mathsf{x})\,\mathrm{d}_\mu u_\nu(\mathsf{x}) = -\mathrm{d}_\mu q^\mu(\mathsf{x}) - \tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu u_\nu(\mathsf{x}),$$

where the second equality follows from conditions (X.39b) and (X.39c), which altogether yield $u_\nu(\mathsf{x})\tau^{\mu\nu}(\mathsf{x}) = -q^\mu(\mathsf{x})$. One thus obtains

$$u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) + \mathrm{d}_\mu q^\mu(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu(\mathsf{x})u_\nu(\mathsf{x}) = 0. \tag{X.46c}$$

Equations (X.46a)–(X.46c) represent the relations governing the dynamics of a dissipative fluid in an arbitrary frame.

Remarks:

∗ The last term on the left hand side of Eq. (X.46c) can be further transformed. First, invoking identity (X.45c) and again $u_\mu(\mathsf{x})\tau^{\mu\nu}(\mathsf{x}) = -q^\nu(\mathsf{x})$, one finds

$$\tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu u_\nu(\mathsf{x}) = q^\nu(\mathsf{x})\mathsf{u}^\rho(\mathsf{x})\mathrm{d}_\rho\mathsf{u}_\nu(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})\mathsf{u}_\nu(\mathsf{x}).$$

Then the terms $u^\mu(\mathsf{x})q^\nu(\mathsf{x})$ and $u^\nu(\mathsf{x})q^\mu(\mathsf{x})$ in $\tau^{\mu\nu}(\mathsf{x})$ do not contribute to the rightmost term: the former, because the "spatial gradient" $\boldsymbol{\nabla}(\mathsf{x})$ is orthogonal to the four-velocity; the latter, because $u^\nu(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) = \frac{1}{2}\nabla_\mu(\mathsf{x})\big[u^\nu(\mathsf{x})u_\nu(\mathsf{x})\big]$ is the derivative of a constant. All in all, one has

$$\tau^{\mu\nu}(\mathsf{x})\,\mathrm{d}_\mu u_\nu(\mathsf{x}) = q^\nu(\mathsf{x})\mathsf{u}^\rho(\mathsf{x})\mathrm{d}_\rho\mathsf{u}_\nu(\mathsf{x}) + \varpi^{\mu\nu}(\mathsf{x})\,\nabla_\mu(\mathsf{x})\mathsf{u}_\nu(\mathsf{x}). \tag{X.47}$$

∗ From the general equations of motion (X.46) one can deduce the simpler forms they take when adopting Eckart's or Landau's choice of four-velocity. With Eckart's definition, the third term $\mathsf{d}\cdot\mathsf{\nu}_a(\mathsf{x})$ in Eq. (X.46a) drops out for the charge current used to define $\mathsf{u}(\mathsf{x})$, since $\mathsf{\nu}_a(\mathsf{x}) = 0$.

In turn, when using Landau's definition,[67] the heat 4-current $\mathsf{q}(\mathsf{x})$ vanishes while $\boldsymbol{\tau}(\mathsf{x})$ reduces to $\boldsymbol{\varpi}(\mathsf{x})$, which leads to simplifications in Eqs. (X.46b) and (X.46c).

Entropy law in a dissipative relativistic fluid

Inserting the thermodynamic relation $\epsilon + \mathcal{P} = Ts + \sum \mu_a n_a$ into the dynamical equation (X.46c) and using $\mathrm{d}\epsilon = T\,\mathrm{d}s + \sum \mu_a\,\mathrm{d}n_a$, one finds

$$T(\mathsf{x})\mathrm{d}_\mu\big[s(\mathsf{x})u^\mu(\mathsf{x})\big] + \sum \mu_a\mathrm{d}_\mu\big[n_a(\mathsf{x})u^\mu(\mathsf{x})\big] + \mathrm{d}_\mu q^\mu(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})\mathrm{d}_\mu u_\nu(\mathsf{x}) = 0.$$

The second term can be transformed with the help of the charge-conservation equation (X.46a): Invoking in addition relation (X.47) for the fourth term yields

$$T(\mathsf{x})\mathrm{d}_\mu\big[s(\mathsf{x})u^\mu(\mathsf{x})\big] = \sum \mu_a(\mathsf{x})\mathrm{d}_\mu\nu_a^\mu(\mathsf{x}) - \mathrm{d}_\mu q^\mu(\mathsf{x}) - \varpi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) - q^\nu(\mathsf{x})\mathsf{u}^\rho(\mathsf{x})\mathrm{d}_\rho\mathsf{u}_\nu(\mathsf{x}).$$

Next we can add two terms inside the square brackets on the left hand side, which partly cancel

[67] This choice of form for $\mathsf{u}(\mathsf{x})$ is often announced as "let us work in the Landau frame", which means that the local rest frame at each point of the fluid is the Landau frame.

the first two terms in the right member:

$$T(\mathsf{x})\mathrm{d}_\mu\left[s(\mathsf{x})u^\mu(\mathsf{x}) - \sum\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\nu_a^\mu(\mathsf{x}) + \frac{q^\mu(\mathsf{x})}{T(\mathsf{x})}\right] = -T(\mathsf{x})\sum\nu_a^\mu(\mathsf{x})\mathrm{d}_\mu\left[\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\right] - \frac{q^\mu(\mathsf{x})}{T(\mathsf{x})}\mathrm{d}_\mu T(\mathsf{x}) - \varpi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) - q^\nu(\mathsf{x})u^\rho(\mathsf{x})\mathrm{d}_\rho u_\nu(\mathsf{x}). \quad \text{(X.48)}$$

The first two terms on the right hand side can be further transformed: since the dissipative four-currents $\mathsf{v}_a(\mathsf{x})$ and $\mathsf{q}(\mathsf{x})$ are orthogonal to the four-velocity, one has the identities $\nu_a^\mu\mathrm{d}_\mu = \nu_a^\mu\nabla_\mu$ and $q^\mu\mathrm{d}_\mu = q^\mu\nabla_\mu$. Then, since the viscous tensor $\boldsymbol{\varpi}$ is symmetric, one has

$$\varpi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) = \frac{1}{2}\varpi^{\mu\nu}(\mathsf{x})\big[\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) + \nabla_\nu(\mathsf{x})u_\mu(\mathsf{x})\big].$$

Using now the decomposition $\varpi^{\mu\nu}(\mathsf{x}) = \pi^{\mu\nu}(\mathsf{x}) + \Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x})$ [Eq. (X.39d)] and

$$\frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) = \frac{1}{2}\left[\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\Delta_{\mu\nu}(\boldsymbol{\nabla}\cdot\mathsf{u})\right] + \frac{1}{3}\Delta_{\mu\nu}(\boldsymbol{\nabla}\cdot\mathsf{u}) \equiv \mathsf{S}_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}(\boldsymbol{\nabla}\cdot\mathsf{u}),$$

where the $\mathsf{S}_{\mu\nu}$ are the components of a traceless tensor[68]—comparing with Eq. (II.17d), this is the rate-of-shear tensor—, while $\boldsymbol{\nabla}\cdot\mathsf{u}$ is the (spatial) 3-divergence of the 4-velocity field, one finds that Eq. (X.48) becomes

$$\mathrm{d}_\mu S^\mu(\mathsf{x}) = -\frac{\pi^{\mu\nu}(\mathsf{x})}{T(\mathsf{x})}\mathsf{S}_{\mu\nu}(\mathsf{x}) - \frac{\Pi(\mathsf{x})}{T(\mathsf{x})}\nabla_\mu(\mathsf{x})u^\mu(\mathsf{x}) - \sum\nu_a^\mu(\mathsf{x})\nabla_\mu\left[\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\right] - q^\mu(\mathsf{x})\left[\frac{\nabla_\mu(\mathsf{x})T(\mathsf{x})}{T(\mathsf{x})^2} + u^\rho(\mathsf{x})\mathrm{d}_\rho u_\mu(\mathsf{x})\right]. \quad \text{(X.49a)}$$

where we have introduced the definition

$$S^\mu(\mathsf{x}) \equiv s(\mathsf{x})u^\mu(\mathsf{x}) - \sum\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\nu_a^\mu(\mathsf{x}) + \frac{q^\mu(\mathsf{x})}{T(\mathsf{x})}. \quad \text{(X.49b)}$$

These are the components of an entropy 4-current $\mathsf{S}(\mathsf{x})$, comprising on the one hand the convective transport of “thermodynamic” entropy—which is the only contribution present in the perfect-fluid case, see Eq. (X.22)—, and on the other hand contributions from the diffusive currents $\mathsf{v}_a(\mathsf{x})$ and $\mathsf{q}(\mathsf{x})$. The term on the left hand side of Eq. (X.49a) is the 4-divergence of this current, and thus describes its local change.

Let Ω be the 4-volume that represents the space-time trajectory of the fluid between an initial and a final times. Integrating Eq. (X.49b) over Ω while using the same reasoning as in § X.1.1 b, one sees that the left member will yield the change in the total entropy of the fluid between these two times. This entropy variation must be positive to ensure that the second law of thermodynamics holds. Accordingly, one requests that the integrand should be positive: $\mathrm{d}_\mu S^\mu(\mathsf{x}) \geq 0$. This requirement can be used to build models for the dissipative currents.

Remark: Equation (X.49b) is actually only correct at first order.

X.4.4 First order dissipative relativistic fluid dynamics

The decompositions (X.36), (X.40) are purely algebraic and do not imply anything regarding the physics of the fluid. Any such assumption involves two distinct elements: an equation of state, relating the energy density ϵ to the thermodynamic pressure $\mathcal{P}$ and the conserved-charge densities n_a; and a set of *constitutive equations* that model the dissipative effects, i.e. the diffusive charge 4-currents $\mathsf{v}_a(\mathsf{x})$, the heat flux density $\mathsf{q}(\mathsf{x})$ and the dissipative stress tensor $\boldsymbol{\tau}(\mathsf{x})$.

[68] In the notation introduced in the remark at the end of § X.4.1, $\mathsf{S}_{\mu\nu} = \nabla_{\langle\mu}u_{\nu\rangle}$.

Several approaches are possible to construct such constitutive equations. A first one would be to compute the conserved-charge 4-currents and energy-momentum tensor starting from an underlying microscopic theory, in particular from a kinetic description of the fluid constituents. Alternatively, one can work directly at the "macroscopic" level, using the various constraints applying to such models.

A first constraint is that the tensorial structure of the various currents should be the correct one: using as building blocks the 4-velocity u, the 4-gradients of the temperature T, the chemical potential μ, and of u, as well as the projector $\boldsymbol{\Delta}$, one writes the possible expressions of the Lorentz-scalar Π, the 4-vectors ν_a and q, and the tensor $\boldsymbol{\pi}$. Another condition is that the second law of thermodynamics should hold, i.e. that when inserting the dissipative currents in Eq. (X.49b), one obtains a 4-divergence of the entropy 4-current that is always positive.

Working in the Landau frame,[69] in which the heat flux density $\mathsf{q}(\mathsf{x})$ vanishes, and assuming a single conserved charge, the simplest—but not the most general—possibility that satisfies all constraints is to request

$$\Pi(\mathsf{x}) = -\zeta(\mathsf{x})\nabla^\mu(\mathsf{x})u_\mu(\mathsf{x}) \tag{X.50a}$$

for the dissipative pressure,

$$\pi^{\mu\nu}(\mathsf{x}) = -\eta(\mathsf{x})\left[\nabla^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \nabla^\nu(\mathsf{x})u^\mu(\mathsf{x}) - \frac{2}{3}\Delta^{\mu\nu}(\mathsf{x})\left[\nabla^\rho(\mathsf{x})u_\rho(\mathsf{x})\right]\right] = -2\eta(\mathsf{x})\mathbf{S}^{\mu\nu}(\mathsf{x}) \tag{X.50b}$$

for the components of the shear stress tensor, and

$$\nu^\mu(\mathsf{x}) = -\kappa(\mathsf{x})\left[\frac{n(\mathsf{x})T(\mathsf{x})}{\epsilon(\mathsf{x})+\mathcal{P}(\mathsf{x})}\right]^2\nabla^\mu(\mathsf{x})\left[\frac{\mu(\mathsf{x})}{T(\mathsf{x})}\right] \tag{X.50c}$$

for the components of the dissipative conserved-charge 4-current. η, ζ, κ are three positive numbers which implicitly depend on the space-time position, inasmuch as they vary with temperature and chemical potential. The first two ones are obviously the shear and bulk viscosity coefficients, respectively, as hinted at by the similarity with the form (III.27f) of the viscous stress tensor of a non-relativistic Newtonian fluid. Accordingly, the equation of motion (X.46b) in which the dissipative stress tensor is substituted by $\varpi^{\mu\nu} = \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}$ with the forms (X.50a), (X.50b) yields a relativistic version of the Navier–Stokes equation.

What is less obvious is that κ in Eq. (X.50c) does correspond to the heat conductivity—which explains why the coefficient in front of the gradient is written in a rather contrived way.

Inserting the dissipative currents (X.50) in the entropy law (X.49b), the latter becomes

$$\mathsf{d}\cdot\mathsf{S}(\mathsf{x}) = \frac{\boldsymbol{\pi}(\mathsf{x}):\boldsymbol{\pi}(\mathsf{x})}{2\eta(\mathsf{x})T(\mathsf{x})} + \frac{\Pi(\mathsf{x})^2}{\zeta(\mathsf{x})T(\mathsf{x})} + \left[\frac{\epsilon(\mathsf{x})+\mathcal{P}(\mathsf{x})}{n(\mathsf{x})T(\mathsf{x})}\right]^2\frac{\nu(\mathsf{x})^2}{\kappa(\mathsf{x})T(\mathsf{x})}. \tag{X.51}$$

Since $\mathsf{n}(\mathsf{x})$ is space-like and all three transport coefficients η, ζ, κ positive, the right hand side of this equation is positive, as it should.

The constitutive equations (X.50) only involve first order terms in the derivatives of velocity, temperature, or chemical potential. In keeping, the theory constructed with such Ansätze is referred to as *first order dissipative fluid dynamics*—which is the relativistic generalization of the set of laws valid for Newtonian fluids.

This simple analogy with the non-relativistic case, together with the fact that only 3 transport coefficients (for the case with a single conserved charge) are needed, makes the "traditional" for-

[69] The corresponding formulae for Π, $\pi^{\mu\nu}$ and q^μ valid in the Eckart frame, in which ν vanishes, can be found e.g. in Sec. 2.4 of Ref. [45].

mulation of first-order dissipative relativistic fluid dynamics à la Landau presented here — or its variant in the Eckart frame — attractive. However the approaches suffer from an issue that does not affect the non-relativistic counterpart. Indeed, it has been shown that many solutions of the relativistic Navier–Stokes(–Fourier) equations in the Landau–Lifshitz or Eckart formulations are unstable against small perturbations [46]. Such disturbances will grow exponentially with time, on a microscopic typical time scale. As a result, the velocity of given modes can quickly exceed the speed of light, which is of course unacceptable in a relativistic theory. In addition, gradients also grow quickly, leading to the breakdown of the small-gradient assumption that implicitly underlies the construction of first-order dissipative fluid dynamics.

Violations of causality actually occur for short-wavelength modes, which from a physical point of view should not be described by fluid dynamics since they involve length scales on which the system is not “continuous”. As such, the issue is more mathematical than physical. These modes do however play a role in numerical computations, so that there is indeed a problem when one is not working with an analytical solution.

However, it was shown in 2019 [47] that there exist formulations of relativistic first-order dissipative fluid dynamics—still based on the gradients of temperature and chemical potentials,[70] as well as the 4-velocity, as in the non-relativistic case—, using more general classes of reference frames, that are causal and stable against small linear perturbations. This finding also holds in the general-relativistic context [48]. However, the corresponding equations involve more transport coefficients than the Landau or Eckart frame formulations, and the reference frames do not have as simple a physical interpretation.

[70] A crucial point is precisely that of the definitions of these notions.