IX.4 Dissipative relativistic fluids

In a dissipative relativistic fluid, the transport of conserved charges and 4-momentum is not only convective—i.e. caused by the fluid motion—, but may also be diffusive, due e.g. to spatial gradients of the flow velocity, the temperature, or the chemical potentials associated with the charges. The description of these new types of transport necessitates the introduction of additional contributions to the charge-number 4-currents and the energy-momentum tensor (§ IX.4.1), that break the local spatial isotropy of the fluid. As a matter of fact, the local rest frame of the fluid is no longer uniquely determined, but there are in general different choices that lead to "simple" expressions for the dynamical quantities (§ IX.4.2).

Adopting one of these particular reference frames as local rest frame, the general dynamical equations describing the motion of a dissipative relativistic fluid are derived in § IX.4.3. Eventually, § IX.4.4 and IX.4.5...

For the sake of brevity, we adopt in this Section a "natural" system of units in which the speed of light c and the Boltzmann constant k_B equal 1.

IX.4.1 Dissipative currents

To account for the additional types of transport present in dissipative fluids, extra terms are added to the charge-number 4-currents and the energy-momentum tensor. Denoting with a subscript perf. the quantities for a perfect fluid, their counterparts in the dissipative case thus read

$$\mathsf{N}_a(\mathsf{x}) = \mathsf{N}_{a,\text{perf.}}(\mathsf{x}) + \boldsymbol{\nu}_a(\mathsf{x}) \quad , \quad \mathbf{T}(\mathsf{x}) = \mathbf{T}_{\text{perf.}}(\mathsf{x}) + \boldsymbol{\tau}(\mathsf{x}) \tag{IX.34a}$$

or equivalently, in terms of the components on a given coordinate system

$$N_a^\mu(\mathsf{x}) = N_{a,\text{perf.}}^\mu(\mathsf{x}) + \nu_a^\mu(\mathsf{x}) \quad , \quad T^{\mu\nu}(\mathsf{x}) = T_{\text{perf.}}^{\mu\nu}(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x}). \tag{IX.34b}$$

In these equations, $\boldsymbol{\nu}_a(\mathsf{x})$ resp. $\boldsymbol{\tau}(\mathsf{x})$ is a 4-vector resp. a symmetric 4-tensor of degree 2, with components $\nu_a^\mu(\mathsf{x})$ resp. $\tau^{\mu\nu}(\mathsf{x}) = \tau^{\nu\mu}(\mathsf{x})$, that represents a dissipative charge-number resp. energy-momentum flux density.

As in the perfect-fluid case, it is natural to introduce a 4-velocity $\mathsf{u}(\mathsf{x})$ in terms of which the quantities $N_{a,\text{perf.}}(\mathsf{x})$, $\mathbf{T}_{\text{perf.}}(\mathsf{x})$ have a simple, "isotropic" expression. Accordingly, let $\mathsf{u}(\mathsf{x})$ be an at first *arbitrary* time-like 4-vector field with constant magnitude -1,[63] with components $u^\mu(\mathsf{x})$, $\mu \in \{0,1,2,3\}$. The reference frame in which the spatial components of this 4-velocity vanish will constitute the local rest frame LR(x) associated with $\mathsf{u}(\mathsf{x})$.

The projector $\boldsymbol{\Delta}$ on the 3-dimensional space orthogonal to the 4-velocity $\mathsf{u}(\mathsf{x})$ is defined as in Eq. (IX.19b), i.e. its components are[63]

$$\Delta^{\mu\nu}(\mathsf{x}) \equiv g^{\mu\nu}(\mathsf{x}) + u^\mu(\mathsf{x})u^\nu(\mathsf{x}), \tag{IX.35}$$

with $g^{\mu\nu}(\mathsf{x})$ the components of the inverse metric tensor $\mathbf{g}^{-1}(\mathsf{x})$. For the comprehension it is important to realize that $\boldsymbol{\Delta}$ plays the role of the identity in the 3-space orthogonal to $\mathsf{u}(\mathsf{x})$.

By analogy with Eqs. (IX.17a), (IX.18), and (IX.19a), one thus writes

$$\boxed{N_a^\mu(\mathsf{x}) = \mathcal{N}_a(\mathsf{x})u^\mu(\mathsf{x}) + \nu_a^\mu(\mathsf{x})} \tag{IX.36a}$$

or equivalently

$$\boxed{\mathsf{N}_a(\mathsf{x}) = \mathcal{N}_a(\mathsf{x})\mathsf{u}(\mathsf{x}) + \boldsymbol{\nu}_a(\mathsf{x})} \tag{IX.36b}$$

and

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \mathcal{E}(\mathsf{x})u^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \mathcal{P}(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})} \tag{IX.37a}$$

i.e., in component-free form

$$\boxed{\mathbf{T}(\mathsf{x}) = \mathcal{E}(\mathsf{x})\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \mathcal{P}(\mathsf{x})\boldsymbol{\Delta}(\mathsf{x}) + \boldsymbol{\tau}(\mathsf{x}).} \tag{IX.37b}$$

Remarks:

∗ The presence of the extra tensor $\boldsymbol{\tau}(\mathsf{x})$ means that the energy-momentum will generally no longer be diagonal in the local rest frame defined by the 4-velocity $\mathsf{u}(\mathsf{x})$, in contrast to the perfect-fluid case [cf. Eq. (IX.16c)].

∗ The careful reader will have noticed that the scalar fields multiplying the four-velocity or the projector $\boldsymbol{\Delta}$ in Eqs. (IX.36)–(IX.37) are denoted with different characters from the perfect-fluid case, Eqs. (IX.17)–(IX.18): $\mathcal{N}_a$, $\mathcal{E}$, $\mathcal{P}$ instead of n_a, ϵ, $\mathscr{P}$. The reason is that the new fields $\mathcal{E}(\mathsf{x})$, $\mathcal{P}(\mathsf{x})$, $\{\mathcal{N}_a(\mathsf{x})\}$ depend on the still arbitrary four-velocity $\mathsf{u}(\mathsf{x})$, and thus it is not guaranteed that they correspond to the quantities (energy density, pressure, number densities) that enter thermodynamic relations.

[63] since $c^2 = 1$

The underlying issue is that the presence of dissipative currents, that lead to non-conservation of entropy, betrays the fact that the system is not at (local) thermodynamic equilibrium. This will be discussed again below, **where?**.

The precise physical content and mathematical form of the additional terms can now be further specified.

Tensor algebra

To simplify the physical interpretation of the constitutive equations (IX.36)–(IX.37), one imposes a few restrictions on the dissipative currents $\boldsymbol{\nu}_a$ and $\boldsymbol{\tau}$.

First, one demands that $\mathcal{N}_a(\mathsf{x})$ should represent the comoving charge density, or equivalently

$$\mathcal{N}_a(\mathsf{x}) = -u_\mu(\mathsf{x})N_a^\mu(\mathsf{x}) = -\mathsf{u}(\mathsf{x})\cdot\mathsf{N}_a(\mathsf{x}), \tag{IX.38}$$

where the minus sign comes from $u_0(\mathsf{x}) = -1$ in the local rest frame. Thus, the dissipative 4-vector $\boldsymbol{\nu}_a(\mathsf{x})$ can have no timelike component in the local rest frame LR(x) defined by the fluid 4-velocity, i.e. the condition

$$u_\mu(\mathsf{x})\nu_a^\mu(\mathsf{x})\Big|_{\mathrm{LR(x)}} = 0$$

should hold in the local rest frame. Since the term on the left hand side of this identity is a Lorentz scalar, its value remains the same in any reference frame or coordinate system, i.e.

$$u_\mu(\mathsf{x})\nu_a^\mu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\boldsymbol{\nu}_a(\mathsf{x}) = 0. \tag{IX.39a}$$

Equations (IX.36) thus represent the decomposition of a 4-vector in a component parallel to the flow 4-velocity and a component orthogonal to it. Accordingly, one can write

$$\nu_a^\mu(\mathsf{x}) = \Delta^\mu{}_\nu(\mathsf{x})N_a^\nu(\mathsf{x}). \tag{IX.39b}$$

Physically, the 4-vector $\boldsymbol{\nu}_a(\mathsf{x})$ represents a *diffusive charge-number 4-current* in the local rest frame, which describes the non-convective transport of the conserved charge of type a.

Similarly, one requests that $\mathcal{E}(\mathsf{x})$ should represent the energy density in the local rest frame, i.e. $\mathcal{E}(\mathsf{x}) = T^{00}(\mathsf{x})\big|_{\mathrm{LR(x)}}$, or equivalently

$$\mathcal{E}(\mathsf{x}) = u_\mu(\mathsf{x})T^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\mathbf{T}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}). \tag{IX.40}$$

That is, the dissipative energy-momentum current $\boldsymbol{\tau}(\mathsf{x})$ can have no 00-component in the local rest frame. This is ensured provided the components $\tau^{\mu\nu}(\mathsf{x})$ do not involve any term proportional to the product $u^\mu(\mathsf{x})u^\nu(\mathsf{x})$. The most general symmetric tensor of degree 2 which satisfies that requirement is of the form

$$\tau^{\mu\nu}(\mathsf{x}) = q^\mu(\mathsf{x})u^\nu(\mathsf{x}) + q^\nu(\mathsf{x})u^\mu(\mathsf{x}) + \varpi^{\mu\nu}(\mathsf{x}), \tag{IX.41a}$$

with $q^\mu(\mathsf{x})$ resp. $\varpi^{\mu\nu}(\mathsf{x})$ the components of a 4-vector $\mathsf{q}(\mathsf{x})$ resp. of a $\binom{2}{0}$-type tensor $\boldsymbol{\varpi}(\mathsf{x})$ such that

$$u_\mu(\mathsf{x})q^\mu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\mathsf{q}(\mathsf{x}) = 0 \tag{IX.41b}$$

and

$$u_\mu(\mathsf{x})\varpi^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\boldsymbol{\varpi}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}) = 0. \tag{IX.41c}$$

Condition (IX.41b) expresses the orthogonality of the 4-velocity $\mathsf{u}(\mathsf{x})$ and the 4-vector $\mathsf{q}(\mathsf{x})$, which physically represents the *heat current* or *energy flux density* in the local rest frame.

In turn, the symmetric tensor $\boldsymbol{\varpi}(\mathsf{x})$ can be decomposed into the sum of a traceless symmetric tensor $\boldsymbol{\varpi}(\mathsf{x})$ with components $\varpi^{\mu\nu}(\mathsf{x})$ and a tensor proportional to the projector (IX.19b) orthogonal to the 4-velocity

$$\varpi^{\mu\nu}(\mathsf{x}) = \pi^{\mu\nu}(\mathsf{x}) + \Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}). \tag{IX.41d}$$

The tensor $\boldsymbol{\pi}(\mathsf{x})$ is the *shear stress tensor* in the local rest frame of the fluid, that describes the transport of momentum due to shear deformations. Eventually, $\Pi(\mathsf{x})$ represents a *dissipative pressure* term, since it behaves as the thermodynamic pressure $\mathcal{P}(\mathsf{x})$ as shown by Eq. (IX.42) below.

All in all, the components of the energy-momentum tensor in a dissipative relativistic fluid may thus be written as

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \mathcal{E}(\mathsf{x})u^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \big[\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})\big]\Delta^{\mu\nu}(\mathsf{x}) + q^\mu(\mathsf{x})u^\nu(\mathsf{x}) + q^\nu(\mathsf{x})u^\mu(\mathsf{x}) + \pi^{\mu\nu}(\mathsf{x}),} \tag{IX.42a}$$

which in geometric formulation reads

$$\boxed{\mathbf{T}(\mathsf{x}) = \mathcal{E}(\mathsf{x})\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \big[\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})\big]\boldsymbol{\Delta}(\mathsf{x}) + \mathsf{q}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \mathsf{u}(\mathsf{x})\otimes\mathsf{q}(\mathsf{x}) + \boldsymbol{\pi}(\mathsf{x}).} \tag{IX.42b}$$

One easily checks the identities

$$q^\mu(\mathsf{x}) = \Delta^{\mu\nu}(\mathsf{x})T_{\nu\rho}(\mathsf{x})u^\rho(\mathsf{x}); \tag{IX.43a}$$

$$\pi^{\mu\nu}(\mathsf{x}) = \frac{1}{2}\bigg[\Delta^\mu{}_\rho(\mathsf{x})\Delta^\nu{}_\sigma(\mathsf{x}) + \Delta^\nu{}_\rho(\mathsf{x})\Delta^\mu{}_\sigma(\mathsf{x}) - \frac{2}{3}\Delta^{\mu\nu}(\mathsf{x})\Delta_{\rho\sigma}(\mathsf{x})\bigg]T^{\rho\sigma}(\mathsf{x}); \tag{IX.43b}$$

$$\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x}) = \frac{1}{3}\Delta^{\mu\nu}(\mathsf{x})T_{\mu\nu}(\mathsf{x}), \tag{IX.43c}$$

which together with Eq. (IX.40) allow one to recover the various fields in which the energy-momentum tensor has been decomposed.

Remarks:

∗ Equations (IX.38) and (IX.40) are often referred to as *Landau matching conditions*.

∗ The energy-momentum tensor comprises 10 unknown independent fields, namely the components $T^{\mu\nu}(\mathsf{x})$ with $\nu \geq \mu$. In the decomposition (IX.42), written in the local rest frame, $\mathcal{E}(\mathsf{x})$, $\mathcal{P}(\mathsf{x})+\Pi(\mathsf{x})$, the spatial components $q^i(\mathsf{x})$ and $\pi^{ij}(\mathsf{x})$ represent 1+1+3+5=10 equivalent independent fields—out of the 6 components $\pi^{ij}(\mathsf{x})$ with $j \geq i$, one of the diagonal ones is fixed by the condition on the trace. This in particular shows that the decomposition of the left hand side of Eq. (IX.43c) into two terms is as yet premature—the splitting actually requires of an equation of state to properly specify $\mathcal{P}(\mathsf{x})$.

Similarly, the 4 unknown components N_a^μ of the 4-current associated with the conserved charge of type a are expressed in terms of $\mathcal{N}_a(\mathsf{x})$ and the three spatial components $\nu_a^i(\mathsf{x})$, i.e. an equivalent number of independent fields.

∗ Let $a^{\mu\nu}$ denote the (contravariant) components of an arbitrary $\binom{2}{0}$-tensor. One encounters in the literature the various notations

$$a^{(\mu\nu)} \equiv \frac{1}{2}\big(a^{\mu\nu} + a^{\nu\mu}\big),$$

which represents the symmetric part of the tensor,

$$a^{[\mu\nu]} \equiv \frac{1}{2}\big(a^{\mu\nu} - a^{\nu\mu}\big)$$

for the antisymmetric part—so that $a^{\mu\nu} = a^{(\mu\nu)} + a^{[\mu\nu]}$—, and

$$a^{\langle\mu\nu\rangle} \equiv \bigg(\Delta^{(\mu}_\rho\Delta^{\nu)}_\sigma - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\rho\sigma}\bigg)a^{\rho\sigma},$$

which is the symmetrized traceless projection on the 3-space orthogonal to the 4-velocity. Using these notations, the dissipative stress tensor (IX.41a) reads

$$\tau^{\mu\nu}(\mathsf{x}) = q^{(\mu}(\mathsf{x})u^{\nu)}(\mathsf{x}) + \pi^{\mu\nu}(\mathsf{x}) + \Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}),$$

while Eq. (IX.43b) becomes $\pi^{\mu\nu}(\mathsf{x}) = T^{\langle\mu\nu\rangle}(\mathsf{x})$.

IX.4.2 Local rest frames

At a given point in a dissipative relativistic fluid, the conserved charge(s) and the energy may flow in different directions. This can happen in particular because particle–antiparticle pairs, which do not contribute to the net charge density, still transport energy. Another, not exclusive, possibility is that different conserved quantum numbers flow in different directions. In any case, one can in general not find a preferred reference frame in which the local properties of the fluid are isotropic.

As a consequence, there is also no unique "natural" choice for the 4-velocity $\mathsf{u}(\mathsf{x})$ of the fluid motion. On the contrary, several definitions of the flow 4-velocity are possible, which imply varying relations for the dissipative currents, although the physics that is being described remains the same.

IX.4.2 a Eckart frame

A first natural possibility, proposed by Eckart[(bs)], is to take the 4-velocity proportional to the 4-current $\mathsf{N}(\mathsf{x})$ for a given conserved charge, namely

$$u^\mu_{\rm Eckart}(\mathsf{x}) \equiv \frac{N^\mu(\mathsf{x})}{\sqrt{-N_\nu(\mathsf{x})N^\nu(\mathsf{x})}}. \tag{IX.44}$$

Accordingly, the dissipative charge-number flux $\boldsymbol{\nu}(\mathsf{x})$ vanishes automatically, so that the expression of charge conservation is simpler with that choice.
The local rest frame associated with the flow 4-velocity (IX.44) is then referred to as *Eckart frame*.

A drawback of that definition of the fluid 4-velocity is that the net charge number can possibly vanish in some regions of a given flow, so that $\mathsf{u}_{\rm Eckart}(\mathsf{x})$ is not defined unambiguously in such domains.

IX.4.2 b Landau frame

An alternative natural definition is that of Landau (and Lifshitz), according to whom the fluid 4-velocity is taken to be proportional to the energy flux density. The corresponding 4-velocity is defined by the implicit equation

$$u^\mu_{\rm Landau}(\mathsf{x}) = \frac{T^\mu{}_\nu(\mathsf{x})\,u^\nu_{\rm Landau}(\mathsf{x})}{\sqrt{-u^\lambda_{\rm Landau}(\mathsf{x})\,T_\lambda{}^\rho(\mathsf{x})\,T_{\rho\sigma}(\mathsf{x})\,u^\sigma_{\rm Landau}(\mathsf{x})}}. \tag{IX.45}$$

With this choice, which in turn determines the *Landau frame*, the heat current $\mathsf{q}(\mathsf{x})$ vanishes, see Eq. (IX.43a), so that the dissipative tensor $\boldsymbol{\tau}(\mathsf{x})$ satisfies the condition

$$u^\mu_{\rm Landau}(\mathsf{x})\,\tau_{\mu\nu}(\mathsf{x}) = 0 \tag{IX.46}$$

and reduces to its "viscous" part $\boldsymbol{\varpi}(\mathsf{x})$.

For a fluid without conserved quantum number, the Landau definition of the 4-velocity is the only natural one. However, in the presence of a conserved charge, heat conduction now enters the dissipative part of the associated current $\mathsf{n}(\mathsf{x})$, which conflicts with the intuition gained in the non-relativistic case. This implies that the Landau choice does not lead to a simple behavior in the limit of low velocities.

Remark: Relation (IX.45) means that the Landau 4-velocity is at every point x an eigenvector of the energy momentum tensor, and more precisely an eigenvector with a negative value — the negative of the local energy density $\epsilon(\mathsf{x})$. This property allows the determination of $\mathsf{u}_{\rm Landau}(\mathsf{x})$.

Eventually, one may of course choose to work with a general fluid 4-velocity $\mathsf{u}(\mathsf{x})$, and thus to keep both the diffusive charge current(s) and the heat flux density in the dynamical fields (IX.36)–(IX.42). We shall come back to that point at the end of § IX.4.4.

[(bs)] C. Eckart, 1902–1973