Chapter IX

Convective heat transfer

The examples of dissipative flows we have seen until now were dominated either by viscous effects (Chap. V and § VI.1.4) or by convective motion (Chap. VIII). In either case, the energy-conservation equation (III.37), and in particular the term representing heat conduction, was never playing a dominant role, with the exception of a brief mention of heat transport in the study of static Newtonian fluids (§ V.1.1).

The purpose of this Chapter is to shift the focus, and to discuss motions of Newtonian fluids in which heat is transferred from one region of the fluid to another. A first such type of transfer is heat conduction, which was already encountered in the static case. Under the generic term "convection", or "convective heat transfer", one encompasses flows in which heat is also transported by the moving fluid, not only via conduction.

Heat transfer will be caused by differences in temperature in a fluid. Going back to the equations of motion, one can make a few assumptions so as to eliminate or at least suppress other effects, and emphasize the role of temperature gradients in moving fluids (Sec IX.1). A specific instance of fluid motion driven by a temperature difference, yet also controlled by the fluid viscosity, which allows for a richer phenomenology, is then presented in Sec. IX.2.

IX.1 Equations of convective heat transfer

The fundamental equations of the dynamics of Newtonian fluids introduced in Chap. III include heat conduction, in the form of a term involving the gradient of temperature, yet the change in time of temperature does not appear explicitly. To obtain an equation involving the time derivative of temperature, some rewriting of the basic equations is thus needed, which will be done together with a few simplifications (§ IX.1.1). Conduction in a static fluid is then recovered as a limiting case.

In many instances, the main effect of temperature differences is however rather to lead to variations of the mass density, which in turn trigger the fluid motion. To have a more adapted description of such phenomena, a few extra simplifying assumptions are made, leading to a new, closed set of coupled equations (§ IX.1.2).

IX.1.1 Basic equations of heat transfer

Consider a Newtonian fluid submitted to conservative volume forces $\vec{f}_V = -\rho\vec{\nabla}\Phi$. Its motion is governed by the laws established in Chap. III, namely by the continuity equation, the Navier–Stokes equation, and the energy-conservation equation or equivalently the entropy-balance equation, which we now recall.

Expanding the divergence of the mass flux density, the continuity equation (III.9) becomes

$$\frac{\mathrm{D}\rho(t,\vec{r})}{\mathrm{D}t} = -\rho(t,\vec{r})\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}). \tag{IX.1a}$$

In turn, the Navier–Stokes equation (III.31a) may be written in the form

$$\rho(t,\vec{r})\frac{\mathrm{D}\vec{\mathrm{v}}(t,\vec{r})}{\mathrm{D}t} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) - \rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}) + 2\vec{\nabla}\cdot\big[\eta(t,\vec{r})\mathbf{S}(t,\vec{r})\big] + \vec{\nabla}\big[\zeta(t,\vec{r})\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]. \tag{IX.1b}$$

Eventually, straightforward algebra using the continuity equation allows one to rewrite the entropy balance equation (III.42b) as

$$\rho(t,\vec{r})\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right] = \frac{1}{T(t,\vec{r})}\vec{\nabla}\cdot\big[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\big] + \frac{2\eta(t,\vec{r})}{T(t,\vec{r})}\mathbf{S}(t,\vec{r}):\mathbf{S}(t,\vec{r}) + \frac{\zeta(t,\vec{r})}{T(t,\vec{r})}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]^2. \tag{IX.1c}$$

Since we wish to isolate effects directly related with the transfer of heat, or playing a role in it, we shall make a few assumptions, so as to simplify the above set of equations.

- The transport coefficients η, ζ, κ depend on the local thermodynamic state of the fluid, i.e. on its local mass density ρ and temperature T, and thereby indirectly on time and position. Nevertheless, they will be taken as constant and uniform throughout the fluid, and pulled out of the various derivatives in Eqs. (IX.1b)–(IX.1c). This is a reasonable assumption as long as only small variations of the fluid properties are considered, which is consistent with the next assumption.

  Somewhat abusively, we shall in fact even allow ourselves to consider η resp. κ as uniform in Eq. (IX.1b) resp. (IX.1c), later replace them by related (diffusion) coefficients $\nu = \eta/\rho$ resp. $\alpha = \kappa/\rho c_{\mathcal{P}}$, and then consider the latter as uniform constant quantities.
  The whole procedure is only "justified" in that one can check—by comparing calculations using this assumption with numerical computations performed without the simplifications—that it does not lead to omitting a physical phenomenon.

- The fluid motions under consideration will be assumed to be "slow", i.e. to involve a small flow velocity, in the following sense:
  - The incompressibility condition $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}) = 0$ will hold on the right hand sides of each of Eqs. (IX.1). Accordingly, Eq. (IX.1a) simplifies to $\mathrm{D}\rho(t,\vec{r})/\mathrm{D}t = 0$ while Eq. (IX.1b) becomes the incompressible Navier–Stokes equation

$$\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r}) = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) - \vec{\nabla}\Phi(t,\vec{r}) + \nu\triangle\vec{\mathrm{v}}(t,\vec{r}), \tag{IX.2}$$

    with a constant and uniform kinematic (shear) viscosity ν.
  - The rate of shear is small, so that its square can be neglected in Eq. (IX.1c). Accordingly, that equation simplifies to

$$\rho(t,\vec{r})T(t,\vec{r})\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right] = \kappa\triangle T(t,\vec{r}). \tag{IX.3}$$

The term on the left hand side of that equation can be further rewritten. As a matter of fact, one can show that the differential of the specific entropy in a fluid particle is related to the change in temperature by

$$T\,\mathrm{d}\left(\frac{s}{\rho}\right) = c_{\mathcal{P}}\,\mathrm{d}T, \tag{IX.4}$$

where $c_{\mathcal{P}}$ denotes the specific heat capacity at constant pressure of the fluid. In turn, this relation translates into an identity relating the material derivatives when the fluid particles are followed in their motion. The left member of Eq. (IX.3) may then be reformulated in terms of the material derivative of the temperature. Introducing the *thermal diffusivity*[(lxxii)]

$$\alpha \equiv \frac{\kappa}{\rho c_{\mathcal{P}}}, \tag{IX.5}$$

which will from now on be assumed to be constant and uniform in the fluid, where $\rho c_{\mathcal{P}}$ is the volumetric heat capacity at constant pressure, one eventually obtains

$$\frac{\mathrm{D}T(t,\vec{r})}{\mathrm{D}t} = \frac{\partial T(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]T(t,\vec{r}) = \alpha \triangle T(t,\vec{r}), \tag{IX.6}$$

which is sometimes referred to as (*convective*) *heat transfer equation*.

If the fluid is at rest or if its velocity is "small" enough to ensure that the convective term $\vec{\mathsf{v}}\cdot\vec{\nabla}T$ remains negligible, Eq. (IX.6) simplifies to the classical heat diffusion equation, with diffusion constant α.

The thermal diffusivity α thus measures the ability of a medium to transfer heat diffusively, just like the kinematic shear viscosity ν characterizes the diffusive transfer of momentum. Accordingly, both coefficients have the same dimension $\mathsf{L}^2\,\mathsf{T}^{-1}$, and can thus be compared meaningfully. Their relative strength is measured by the dimensionless *Prandtl number*

$$\mathrm{Pr} \equiv \frac{\nu}{\alpha} = \frac{\eta c_{\mathcal{P}}}{\kappa} \tag{IX.7}$$

which in contrast to the Mach, Reynolds, Froude, Ekman, Rossby... numbers encountered in the previous chapters is entirely determined by the fluid, independent of any flow characteristics.

IX.1.2 Boussinesq approximation

If there is a temperature gradient in a fluid, it will lead to a heat flux density, and thereby to the transfer of heat, thus influencing the fluid motion. However, heat exchanges by conduction are often slow—except in metals—, so that another effect due to temperature differences is often the first to play a significant role, namely thermal expansion (or contraction), which will lead to buoyancy (§ IV.1.1) when a fluid particle acquires a mass density different from that of its surroundings.

The simplest approach to account for this effect, due to Boussinesq,[(55)] consists in considering that even though the fluid mass density changes, nevertheless the motion can to a very good approximation be viewed as incompressible—which is what was assumed in § IX.1.1:

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) \simeq 0, \tag{IX.8}$$

where $\simeq$ is used to allow for small relative variations in the mass density, which are directly related to the expansion rate through Eq. (IX.1a).

Denoting by T_0 a typical temperature in the fluid and ρ_0 the corresponding mass density (strictly speaking, at a given pressure), the effect of thermal expansion on the latter reads

$$\rho(\Theta) = \rho_0(1 - \alpha_{(\mathcal{V})}\Theta), \tag{IX.9}$$

with

$$\Theta \equiv T - T_0 \tag{IX.10}$$

the temperature difference measured with respect to the reference value, and

$$\alpha_{(\mathcal{V})} \equiv -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{\mathcal{P},N} \tag{IX.11}$$

the *thermal expansion coefficient* for volume, where the derivative is taken at the thermodynamic

(55) Hence its denomination *Boussinesq approximation* (for buoyancy).

(lxxii) *Temperaturleitfähigkeit*

point corresponding to the reference value ρ_0. Strictly speaking, the linear regime (IX.9)[56] only holds when $|\alpha_{(\mathcal{V})}\Theta| \ll 1$, as will be assumed hereafter.

Consistent with relation (IX.9), the pressure term in the incompressible Navier–Stokes equation can be approximated as

$$-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) \simeq -\frac{\vec{\nabla}\mathcal{P}(t,\vec{r})}{\rho_0}\big[1+\alpha_{(\mathcal{V})}\Theta(t,\vec{r})\big].$$

Introducing an effective pressure $\mathcal{P}_{\text{eff}}$ that accounts for the leading effect of the potential Φ from which the volume forces derive,

$$\mathcal{P}_{\text{eff.}}(t,\vec{r}) \equiv \mathcal{P}(t,\vec{r}) + \rho_0\Phi(t,\vec{r}),$$

one finds

$$-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) - \vec{\nabla}\Phi(t,\vec{r}) \simeq -\frac{\vec{\nabla}\mathcal{P}_{\text{eff.}}(t,\vec{r})}{\rho_0} + \alpha_{(\mathcal{V})}\Theta(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}),$$

where a term of subleading order $\alpha_{(\mathcal{V})}\Theta\vec{\nabla}\mathcal{P}_{\text{eff.}}$ has been dropped. To this level of approximation, the incompressible Navier–Stokes equation (IX.2) becomes

$$\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}) = -\frac{\vec{\nabla}\mathcal{P}_{\text{eff.}}(t,\vec{r})}{\rho_0} + \alpha_{(\mathcal{V})}\Theta(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}) + \nu\triangle\vec{\mathsf{v}}(t,\vec{r}). \tag{IX.12}$$

This form of the Navier–Stokes equation emphasizes the role of a finite temperature difference Θ in providing an extra force density which contributes to the buoyancy, supplementing the effective pressure term.

Eventually, definition (IX.10) together with the convective heat transfer equation (IX.6) lead at once to

$$\frac{\partial\Theta(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\Theta(t,\vec{r}) = \alpha\triangle\Theta(t,\vec{r}). \tag{IX.13}$$

The (*Oberbeck*[at]–)*Boussinesq equations* (IX.8), (IX.12), and (IX.13) represent a closed system of five coupled scalar equations for the dynamical fields $\vec{\mathsf{v}}$, Θ—which in turn determines the variation of the mass density—and $\mathcal{P}_{\text{eff.}}$.

IX.2 Rayleigh–Bénard convection

A relatively simple example of flow in which thermal effects play a major role is that of a fluid with a positive[57] thermal expansion coefficient $\alpha_{(\mathcal{V})}$ between two horizontal plates at constant but different temperatures, the lower plate being at the higher temperature, in a uniform gravitational potential $-\vec{\nabla}\Phi(t,\vec{r}) = \vec{g}$, in the absence of horizontal pressure gradient.

The distance between the two plates will be denoted by d, and the temperature difference between them by ΔT, with $\Delta T > 0$ when the lower plate is warmer. When needed, a system of Cartesian coordinates will be used, with the (x, y)-plane midway between the plates and a vertical z-axis, with the acceleration of gravity pointing towards negative values of z, i.e. $\Phi(t,\vec{r}) = gz$.

IX.2.1 Phenomenology of the Rayleigh–Bénard convection

IX.2.1 a Experimental findings

If both plates are at the same temperature or if the upper one is warmer ($\Delta T < 0$), the fluid between them can simply be at rest, with a stationary linear temperature profile.

[56]... which is in fact the beginning of a Taylor expansion in Θ.

[57]This is the "usual" behavior, but there famously exist fluids with a density anomaly, like liquid water below 4°C.

[at]A. Oberbeck, 1849–1900

As a matter of fact, denoting by T_0 resp. $\mathcal{P}_0$ the temperature resp. pressure at a point at $z = 0$ and ρ_0 the corresponding mass density, one easily checks that equations (IX.8), (IX.12), (IX.13) admit the static solution

$$\vec{v}_{\rm st.}(t,\vec{r}) = \vec{0}, \quad \Theta_{\rm st.}(t,\vec{r}) = -\frac{z}{d}\Delta T, \quad \mathcal{P}_{\rm eff.,st.}(t,\vec{r}) = \mathcal{P}_0 - \rho_0 g\frac{z^2}{2d}\alpha_{(\mathcal{V})}\Delta T, \tag{IX.14}$$

with the pressure given by $\mathcal{P}_{\rm st.}(t,\vec{r}) = \mathcal{P}_{\rm eff.,st.}(t,\vec{r}) - \rho_0 gz$. Since $|z/d| < \frac{1}{2}$ and $|\alpha_{(\mathcal{V})}\Delta T| \ll 1$, one sees that the main part of the pressure variation due to gravity is already absorbed in the definition of the effective pressure.

If $\Delta T = 0$, one recognizes the usual linear pressure profile of a static fluid at constant temperature in a uniform gravity field.

One can check that the fluid state defined by the profile (IX.14) is stable against small perturbations of any of the dynamical fields. To account for that property, that state (for a given temperature difference ΔT) will be referred to as "equilibrium state".

Increasing now the temperature of the lower plate with respect to that of the upper plate, for small positive temperature differences ΔT nothing happens, and the static solution (IX.14) still holds—and is still stable.

When ΔT reaches a critical value ΔT_c, the fluid starts developing a pattern of somewhat regular cylindrical domains rotating around their longitudinal, horizontal axes, two neighboring regions rotating in opposite directions. These domains in which warmer and thus less dense fluid rises on the one side while colder, denser fluid descends on the other side, are called *Bénard cells*.[(au)]

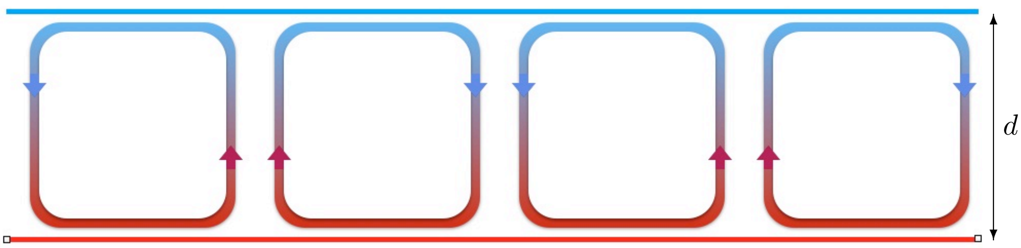

Figure IX.1 – Schematic representation of Bénard cells between two horizontal plates.

The transition from a situation in which the static fluid is a stable state, to that in which motion develops—i.e. the static case is no longer stable—, is referred to as (onset of the) *Rayleigh–Bénard instability*. Since the motion of the fluid appears spontaneously, without the need to impose any external pressure gradient, it is an instance of *free convection* or *natural convection*—in opposition to *forced convection).*

Remarks:

∗ Such convection cells are omnipresent in Nature, as e.g. in the Earth mantle, in the Earth atmosphere, or in the Sun convective zone.

∗ When ΔT further increases, the structure of the convection pattern becomes more complicated, eventually becoming chaotic.

In a series of experiments with liquid helium or mercury, A. Libchaber[(av)] and his collaborators observed the following features [37, 38, 39]: Shortly above ΔT_c, the stable fluid state involve cylindrical convective cells with a constant profile. Above a second threshold, "oscillatory convection" develops: that is, undulatory waves start to propagate along the "surface" of the convective cells, at first at a unique (angular) frequency ω_1, then—as ΔT further increases—also at higher harmonics $n_1\omega_1$, $n_1 \in \mathbb{N}$. As the temperature difference ΔT reaches a third threshold, a second undulation frequency ω_2 appears, incommensurate with ω_1, later accompanied by the combina-

[(au)] H. Bénard, 1874–1939. [(av)] A. Libchaber, born 1934 tions $n_1\omega_1 + n_2\omega_2$, with $n_1, n_2 \in \mathbb{N}$. At higher ΔT, the oscillator with frequency ω_2 experiences a shift from its proper frequency to a neighboring submultiple of ω_1—e.g., $\omega_1/2$ in the experiments with He—, illustrating the phenomenon of *frequency locking*. For even higher ΔT, submultiples of ω_1 appear ("frequency demultiplication"), then a low-frequency continuum, and eventually chaos.

Each appearance of a new frequency may be seen as a *bifurcation*. The ratios of the experimentally measured lengths of consecutive intervals between successive bifurcations provide an estimate of the (first) *Feigenbaum constant*[aw] in agreement with its theoretical value—thereby providing the first empirical confirmation of Feigenbaum's theory.

IX.2.1 b Qualitative discussion

Consider the fluid in its "equilibrium" state of rest, in the presence of a positive temperature difference ΔT, so that the lower layers of the fluid are warmer than the upper ones.

If a fluid particle at altitude z acquires, for some reason, a temperature that differs from the equilibrium temperature—measured with respect to some reference value—$\Theta(z)$, then its mass density given by Eq. (IX.9) will differ from that of its environment. As a result, the Archimedes force acting on it no longer exactly balances its weight, so that it will experience a buoyancy force. For instance, if the fluid particle is warmer that its surroundings, it will be less dense and experience a force directed upwards. Consequently, the fluid particle should start to move in that direction, in which case it encounters fluid which is even colder and denser, resulting in an increased buoyancy and a continued motion. According to that reasoning, *any* vertical temperature gradient should result in a convective motion.

There are however two effects that counteract the action of buoyancy, and explain why the Rayleigh–Bénard instability necessitates a temperature difference larger than a given threshold. First, the rising particle fluid will also experience a viscous friction force from the other fluid regions it passes through, which slows its motion. Secondly, if the rise of the particle is too slow, heat has time to diffuse—by heat conduction—through its surface: this tends to equilibrate the temperature of the fluid particle with that of its surroundings, thereby suppressing the buoyancy.

Accordingly, we can expect to find that the Rayleigh–Bénard instability will be facilitated when $\alpha_{(\mathcal{V})}\Delta T g$—i.e. the buoyancy per unit mass—increases, as well as when the thermal diffusivity α and the shear viscosity ν decrease.

Translating the previous argumentation in formulas, let us consider a spherical fluid particle with radius R, and assume that it has some vertically directed velocity v, while its temperature initially equals that of its surroundings.

With the fluid particle surface area, proportional to R^2, and the thermal diffusivity α, one can estimate the characteristic time for heat exchanges between the particle and the neighboring fluid, namely

$$\tau_Q = C\frac{R^2}{\alpha}$$

with C a geometrical factor. If the fluid particle moves with constant velocity v during that duration τ_Q, while staying at almost constant temperature since heat exchanges remain limited, the temperature difference $\delta\Theta$ it acquires with respect to the neighboring fluid is

$$\delta\Theta = \frac{\partial\Theta}{\partial z}\delta z = \frac{\partial\Theta}{\partial z}v\tau_Q = C\frac{\Delta T}{d}\frac{R^2}{\alpha}v,$$

where $\Delta T/d$ is the vertical temperature gradient in the fluid imposed by the two plates. This temperature difference gives rise to a mass density difference

$$\delta\rho = -\rho_0\alpha_{(\mathcal{V})}\delta\Theta = -C\rho_0 v\frac{R^2}{\alpha}\frac{\alpha_{(\mathcal{V})}\Delta T}{d},$$

[aw] M. Feigenbaum, 1944–2019

between the particle and its surroundings. As a result, the fluid particle experiences an upwards directed buoyancy

$$-\frac{4\pi}{3}R^3\delta\rho\, g = \frac{4\pi C}{3}\rho_0 g v\frac{R^5}{\alpha}\frac{\alpha_{(\mathcal{V})}\Delta T}{d}. \tag{IX.15}$$

On the other hand, the fluid particle is slowed in its vertical motion by the downwards oriented Stokes friction force acting on it, namely, in projection on the z-axis

$$F_{\text{Stokes}} = -6\pi R\eta v. \tag{IX.16}$$

Note that assuming that the velocity v remains constant, with a counteracting Stokes force that is automatically the "good" one, relies on the picture that viscous effects adapt instantaneously, i.e. that momentum diffusion is fast. That is, the above reasoning actually assumes that the Prandtl number (IX.7) is much larger than 1; yet its result is independent from that assumption.

Comparing Eqs. (IX.15) and (IX.16), buoyancy will overcome friction, and thus the Rayleigh–Bénard instability take place, when

$$\frac{4\pi C}{3}\rho_0 g v\frac{R^5}{\alpha}\frac{\alpha_{(\mathcal{V})}\Delta T}{d} > 6\pi R\rho_0\nu v \quad\Leftrightarrow\quad \frac{\alpha_{(\mathcal{V})}\Delta T\, g R^4}{\alpha\nu d} > \frac{9}{2C}.$$

Note that the velocity v that was invoked in the reasoning actually drops out from this condition. Taking for instance $R = d/2$—which maximizes the left member of the inequality—, this becomes

$$\text{Ra} \equiv \frac{\alpha_{(\mathcal{V})}\Delta T\, g\, d^3}{\nu\alpha} > \frac{72}{C} = \text{Ra}_c.$$

Ra is the so-called *Rayleigh number* and Ra_c its critical value, above which the static-fluid state is unstable against perturbation and convection takes place. The "value" $72/C$ found with the above simple reasoning on force equilibrium is totally irrelevant—both careful experiments and theoretical calculations agree with $\text{Ra}_c = 1708$ for a fluid between two very large plates—, the important lesson is the existence of a threshold.

IX.2.2 Toy model for the Rayleigh–Bénard instability

A more refined—although still crude—toy model of the transition to convection consists in considering small perturbations $\vec{\mathsf{v}}$, $\delta\Theta$, $\delta\mathcal{P}_{\text{eff.}}$ around a static state $\vec{\mathsf{v}}_{\text{st.}} = \vec{0}$, $\Theta_{\text{st.}}$, $\mathcal{P}_{\text{eff.,st.}}$, and to linearize the Boussinesq equations to first order in these perturbations. As shown by Eq. (IX.14), the effective pressure $\mathcal{P}_{\text{eff.,st.}}$ actually already includes a small correction, due to $\alpha_{(\mathcal{V})}\Delta T$ being much smaller than 1, so that we may from the start neglect $\delta\mathcal{P}_{\text{eff.}}$.

To first order in the perturbations, Eqs. (IX.12), projected on the z-axis, and (IX.13) give, after subtraction of the contributions from the static solution

$$\frac{\partial \mathsf{v}_z}{\partial t} = \nu\triangle\mathsf{v}_z + \alpha_{(\mathcal{V})}\delta\Theta\, g, \tag{IX.17a}$$

$$\frac{\partial\delta\Theta}{\partial t} - \frac{\Delta T}{d}\mathsf{v}_z = \alpha\triangle\delta\Theta. \tag{IX.17b}$$

Moving the second term of the latter equation to the right hand side increases the parallelism of this set of coupled equations. In addition, there is also the projection of Eq. (IX.12) along the x-axis, and the velocity field must obey the incompressibility condition (IX.8).

The proper approach would now be to specify the boundary conditions, namely: the vanishing of v_z at both plates—impermeability condition—, the vanishing of v_x at both plates—no-slip condition—, and the identity of the fluid temperature at each plate with that of the corresponding plate; that is, all in all, 6 conditions. By manipulating the set of equations, it can be turned into a 6th-order linear partial differential equation for $\delta\Theta$, on which the boundary conditions can be imposed.

Instead of following this long road,[58] we refrain from trying to really solve the equations, but rather make a simple ansatz, namely $\mathsf{v}_z(t,\vec{r}) = \mathsf{v}_0\,\mathrm{e}^{\gamma t}\cos(kx)$—which automatically fulfills the incompressibility equation, but clearly violates the impermeability conditions—, and a similar one for $\delta\Theta$, with γ a constant. Substituting these forms in Eqs. (IX.17) yields the linear system

$$\gamma\mathsf{v}_0 = -k^2\nu\mathsf{v}_0 + \alpha_{(\mathcal{V})}\delta\Theta_0 g \quad\Leftrightarrow\quad (\gamma+\nu k^2)\mathsf{v}_0 - g\alpha_{(\mathcal{V})}\delta\Theta_0 = 0,$$
$$\gamma\delta\Theta_0 = -k^2\alpha\delta\Theta_0 + \frac{\Delta T}{d}\mathsf{v}_0 \quad\Leftrightarrow\quad \frac{\Delta T}{d}\mathsf{v}_0 - (\gamma+\alpha k^2)\delta\Theta_0 = 0$$

for the amplitudes v_0, $\delta\Theta_0$. This admits a non-trivial solution only if

$$(\gamma+\nu k^2)(\gamma+\alpha k^2) - \frac{\alpha_{(\mathcal{V})}\Delta T}{d}g = 0. \tag{IX.18}$$

This is a straightforward quadratic equation for γ. It always has two real solutions, one of which is negative—corresponding to a dampened perturbation—since their sum is $-(\alpha+\nu)k^2 < 0$; the other solution may change sign since their product

$$\alpha\nu k^4 - \frac{\alpha_{(\mathcal{V})}\Delta T}{d}g$$

is positive for $\Delta T = 0$, yielding a second negative solution, yet changes sign as ΔT increases. The vanishing of this product thus signals the onset of instability. Taking for instance $k = \pi/d$ to fix ideas, this occurs at a critical Rayleigh number

$$\mathrm{Ra}_c = \frac{\alpha_{(\mathcal{V})}\Delta T\, g\, d^3}{\alpha\nu} = \pi^4,$$

where the precise value (here π^4) is irrelevant.

From Eq. (IX.18) also follows that the growth rate of the instability is given in the neighborhood of the threshold by

$$\gamma = \frac{\mathrm{Ra}-\mathrm{Ra}_c}{\mathrm{Ra}_c}\frac{\alpha\nu}{\alpha+\nu}k^2,$$

i.e. it is infinitely slow at Ra_c. This is reminiscent of a similar behavior in the vicinity of the critical point associated with a thermodynamic phase transition.

By performing a more rigorous calculation including non-linear effects, one can show that the velocity amplitude at a given point behaves like

$$\mathsf{v} \propto \left(\frac{\mathrm{Ra}-\mathrm{Ra}_c}{\mathrm{Ra}_c}\right)^{\beta} \quad\text{with}\quad \beta = \frac{1}{2} \tag{IX.19}$$

in the vicinity of the critical value, and this prediction is borne out by experiments [41]. Such a power law behavior is again reminiscent of the thermodynamics of phase transitions, more specifically here—since v vanishes below Ra_c and is finite above—of the behavior of the *order parameter* in the vicinity of a critical point. Accordingly the notation β used for the exponent in relation (IX.19) is the traditional choice for the critical exponent associated with the order parameter of phase transitions.

Eventually, a last analogy with phase transitions regards the breaking of a symmetry at the threshold for the Rayleigh–Bénard instability. Below Ra_c, the system is invariant under translations parallel to the plates, while above Ra_c that symmetry is spontaneously broken.

[58] The reader may find details in Ref. [40, Chap. II].

Bibliography for Chapter IX

- A nice introduction to the topic is to be found in Ref. [42], which is a popular science account of part of Ref. [43].
- Faber [1] Chapter 8.5–8.7 & 9.2.
- Guyon *et al.* [2] Chapter 11.2.
- Landau–Lifshitz [4, 5] Chapter V § 49–53 & 56–57.