Chapter VIII

Turbulence in non-relativistic fluids

All examples of flows considered until now in these notes, either of perfect or Newtonian fluids (Chapters IV–VI), shared a common property, namely they were all assumed to be laminar. This assumption—which at once translates into a relative simplicity of the flow velocity profile—is however not the generic case in real flows, which are most often turbulent to a more or less large extent. The purpose of this Chapter is to provide an introduction to the problematic of turbulence in non-relativistic fluid motions.

A number of experiments, in particular those conducted by O. Reynolds, have hinted at the possibility that turbulence occurs when the Reynolds number (V.12) is large enough in the flow, i.e. when convective effects predominate over the shear viscous ones in the mean fluid motion over which the instabilities develop. This distinction between mean flow and turbulent fluctuations can be modeled directly by splitting the dynamical fields into two parts, and one recovers with the help of dimensional arguments the role of the Reynolds number in separating two regimes, one in which viscous effects dominate the mean flow, and one in which turbulence takes over (Sec. VIII.1).

Despite its appeal, the decomposition into a mean flow and a turbulent motion has the drawback that it leads to a system of equations of motion which is not closed. A possibility to remedy this issue is to invoke the notion of a turbulent viscosity, for which various models have been proposed (Sec. VIII.2).

Even when the system of equations of motion is closed, it still involves averages—with an a priori unknown underlying probability distribution. That is, the description of turbulent part of the motion necessitates the introduction of a few concepts characterizing the statistics of the velocity field (Sec. VIII.3).

For the sake of simplicity, we shall mostly consider turbulence in the three-dimensional incompressible motion of Newtonian fluids with constant and homogeneous properties (mass density, viscosity. . .), in the absence of relevant external bulk forces, and neglecting possible temperature gradients—and thereby convective heat transport.

VIII.1 Generalities on turbulence in fluids

In this Section, a few experimental facts on turbulence in fluid flows are presented, and the first steps towards a modeling of the phenomenon are introduced.

VIII.1.1 Phenomenology of turbulence

VIII.1.1 a Historical example: Hagen–Poiseuille flow

The idealized Hagen–Poiseuille flow of a Newtonian fluid in a cylindrical tube was already partly discussed in § V.1.4. There, it was found that in the stationary *laminar* regime in which the velocity field $\vec{\mathsf{v}}$ is everywhere parallel to the walls of the tube, the mass flow rate Q across the cylinder cross section is given by the Hagen–Poiseuille law

$$Q = -\frac{\pi\rho a^4}{8\eta}\frac{\delta\mathcal{P}}{L}, \tag{V.9}$$

with a the tube radius, $\delta\mathcal{P}/L$ the pressure drop per unit length, and ρ, η the fluid mass density and shear viscosity, respectively.

Due to the viscous friction forces, part of the kinetic energy of the fluid motion is transformed into heat. To compensate for these "losses" and keep the flow in the stationary regime, energy has to be provided to the fluid, namely in the form of the mechanical work of the pressure forces driving the flow. Thus, the rate of energy dissipation per unit fluid mass is[45]

$$\dot{\mathcal{E}}_{\rm diss.} = -\frac{1}{\rho}\frac{\delta\mathcal{P}}{L}\langle\mathsf{v}\rangle = \frac{8\nu\langle\mathsf{v}\rangle^2}{a^2} \tag{VIII.1}$$

with $\langle\mathsf{v}\rangle$ the average flow velocity across the tube cross section,

$$\langle\mathsf{v}\rangle = \frac{Q}{\pi a^2\rho} = -\frac{a^2}{8\eta}\frac{\delta\mathcal{P}}{L}.$$

In the laminar regime the rate $\dot{\mathcal{E}}_{\rm diss.}$ is thus proportional to the kinematic viscosity $\nu \equiv \eta/\rho$ and to the square of the average velocity.

According to the Hagen–Poiseuille law (V.9), at fixed pressure gradient the average velocity $\langle\mathsf{v}\rangle$ grows quadratically with the tube radius. In practice, the rise is actually slower, reflecting a higher rate of energy loss in the flow than the laminar prediction (VIII.1). Thus, the mean rate of energy dissipation is no longer proportional to $\langle\mathsf{v}\rangle^2$, but rather to a higher power of $\langle\mathsf{v}\rangle$. Besides, the flow velocity profile across the tube cross section is no longer parabolic, but (in average) flatter around the cylinder axis, with a faster decrease at the tube walls.

VIII.1.1 b Transition to a turbulent regime

Consider a given geometry—say for instance that of the Hagen–Poiseuille flow or the motion of a fluid in a tube with fixed rectangular cross section. In the low-velocity regime, the flow in that geometry is laminar, and the corresponding state[46] is stable against small perturbations, which are damped by viscosity (see § VI.1.4).

However, when the average flow velocity exceeds some critical value, while all other characteristics of the flow, in particular the fluid properties, remain fixed, the motion cannot remain laminar. Small perturbations are no longer damped, but can grow by extracting kinetic energy from the "main", regular part of the fluid motion. As a consequence, instead of simple pathlines, the fluid particles now follow more twisted ones: the flow becomes *turbulent*.

[45] In this Chapter, we shall only discuss incompressible flows at constant mass density ρ, and thus always consider energies per unit mass.

[46] This term really refers to a macroscopic "state" of the system in the statistical-physical sense. In contrast to the global equilibrium states usually considered in thermostatics, it is here a non-equilibrium steady state, in which local equilibrium holds at every point.

In that case, the velocity gradients involved in the fluid motion are on average much larger than in a laminar flow. The amount of viscous friction per unit volume or unit mass of the fluid is thus increased, and a larger fraction of the kinetic energy is dissipated as heat.

The role of a critical parameter in the onset of turbulence was discovered by Reynolds in the case of the Hagen–Poiseuille flow of water, in which he injected some colored water on the axis of the tube, repeating the experiment for increasing flow velocities [21]. In the laminar regime found at small velocities, the streakline formed by the colored water forms a thin band along the tube axis, which does not mix with the surrounding water. Above some flow velocity, the streakline remains straight along some distance in the tube, then suddenly becomes unstable and fills the whole cross section of the tube.

As Reynolds understood himself by performing his experiments with tubes of various diameters, the important parameter is not the velocity itself, but rather the Reynolds number Re (V.12), which is proportional to the velocity. Thus, the transition to turbulence in flows with shear occurs at a "critical value" Re_c, which however depends on the geometry of the flow. For instance Re_c is of order 2000 for the Hagen–Poiseuille flow, but becomes of order 1000 for the plane Poiseuille flow investigated in § V.1.3, while $\mathrm{Re}_c \simeq 370$ for the plane Couette flow (§ V.1.2).

The notion of a critical Reynolds number separating the laminar and turbulent regimes is actually a simplification. In theoretical studies of the stability of the laminar regime against *linear* perturbations, such a critical value Re_c can be computed for some very simple geometries, yielding e.g. $\mathrm{Re}_c = 5772$ for the plane Poiseuille flow. Yet the stability sometimes also depends on the size of the perturbation: the larger it is, the smaller the associated critical Re_c is, which hints at the role of nonlinear instabilities.

In the following, we shall leave aside the problem of the temporal onset of turbulence—and thereby of the (in)stability of laminar flows—, and focus on flows in which turbulence is already established.

VIII.1.2 Reynolds decomposition of the fluid dynamical fields

Experiment as well as reasoning hint at the existence of an underlying "simple", laminar flow over which turbulence develops. Accordingly, a reasonable ansatz for the description of the turbulent motion of a fluid is to split the relevant dynamical fields into two components: a first one that varies slowly both in time t and position $\vec{r}$, and a rapidly fluctuating component, which will be denoted with primed quantities. In the case of the flow velocity field $\vec{\mathsf{v}}(t,\vec{r})$, this *Reynolds decomposition*[(lxii)] reads [29]

$$\vec{\mathsf{v}}(t,\vec{r}) = \overline{\vec{\mathsf{v}}(t,\vec{r})} + \vec{\mathsf{v}}'(t,\vec{r}), \tag{VIII.2}$$

with $\overline{\vec{\mathsf{v}}}$ resp. $\vec{\mathsf{v}}'$ the "slow" resp. "fast" component. For the pressure, one similarly writes

$$\mathcal{P}(t,\vec{r}) = \overline{\mathcal{P}(t,\vec{r})} + \mathcal{P}'(t,\vec{r}). \tag{VIII.3}$$

The fluid motion with velocity $\overline{\vec{\mathsf{v}}}$ and pressure $\overline{\mathcal{P}}$ is then referred to as "mean flow", that with the rapidly varying quantities as "fluctuating motion".

Remarks:

∗ The "fast" and "short wavelength" degrees of freedom that constitute the turbulent motion should still be "slow enough" to be fluid dynamical, i.e. the corresponding scales are still "macroscopic" in the sense introduced in Sec. I.1.

(lxii) *Reynolds-Zerlegung*

∗ Let us already emphasize that the mean flow is *not* a valid solution of the usual fluid-dynamical equations of motion, in particular of the Navier–Stokes equation. Accordingly, the Reynolds decompositions (VIII.2)–(VIII.3) differ from the decomposition into a “background flow” and a “perturbation” introduced in the study of sound waves (Sec. VI.1).

As hinted at by the notation, $\overline{\vec{v}(t,\vec{r})}$ represents an average, with some underlying probability distribution.

Theoretically, the Reynolds average $\overline{\,\cdot\,}$ should be an ensemble average, obtained from an infinitely large number of realizations, namely experiments or computer simulations; in practice, however, there is only a finite number N of realizations $\vec{v}^{(n)}(t,\vec{r})$, where $n = 1, 2, \ldots, N$ labels the realization. If the turbulent flow is statistically stationary, one may invoke the *ergodic assumption* and replace the ensemble average by a time average:

$$\overline{\vec{v}}(\vec{r}) \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \vec{v}^{(n)}(t,\vec{r}) \approx \frac{1}{\mathcal{T}}\int_{t-\mathcal{T}/2}^{t+\mathcal{T}/2} \vec{v}(t',\vec{r})\,\mathrm{d}t',$$

where the duration $\mathcal{T}$ should be much larger than the autocorrelation time of the turbulent velocity $\vec{v}'(t,\vec{r})$. If the flow is not statistically stationary, so that $\overline{\vec{v}}(t,\vec{r})$ also depends on time, then $\mathcal{T}$ must also be much smaller than the typical time scale of the variations of the mean flow.

Using the same averaging procedure, the fluctuating velocity must obey the condition

$$\overline{\vec{v}'(t,\vec{r})} = \vec{0}. \tag{VIII.4}$$

That is, the random variable $\vec{v}'(t,\vec{r})$ is centered for every t and $\vec{r}$.

Despite this fact, the turbulent velocity $\vec{v}'(t,\vec{r})$ still plays a role in the dynamics, in particular that of the mean flow, because its two-point, three-point and higher (auto)correlation functions are in general non-zero. For instance, one can write—assuming that the mass density ρ is constant and uniform throughout the fluid

$$\rho\,\overline{\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})} = \rho\,\overline{\mathsf{v}^i(t,\vec{r})}\;\overline{\mathsf{v}^j(t,\vec{r})} + \rho\,\overline{\mathsf{v}'^i(t,\vec{r})\,\mathsf{v}'^j(t,\vec{r})}.$$

The first term of the right member corresponds to the convective part of the momentum-flux density of the mean flow, while the second one

$$\mathbf{T}_{\mathrm{R}}^{ij}(t,\vec{r}) \equiv \rho\,\overline{\mathsf{v}'^i(t,\vec{r})\,\mathsf{v}'^j(t,\vec{r})}, \tag{VIII.5}$$

which is the ij-component of the rank 2 tensor

$$\boxed{\mathbf{T}_{\mathrm{R}}(t,\vec{r}) \equiv \rho\,\overline{\vec{v}'(t,\vec{r}) \otimes \vec{v}'(t,\vec{r})},} \tag{VIII.6}$$

is due to the rapidly fluctuating motion. $\mathbf{T}_{\mathrm{R}}$ is called *turbulent stress* or *Reynolds stress*.[(lxiii)]

VIII.1.3 Dynamics of the mean flow

For the sake of simplicity, the fluid motion will from now on be assumed to be incompressible. Thanks to the linearity of the averaging process, the kinematic condition $\vec{\nabla}\cdot\vec{v}(t,\vec{r}) = 0$ leads to the two relations

$$\vec{\nabla}\cdot\overline{\vec{v}}(t,\vec{r}) = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{v}'(t,\vec{r}) = 0. \tag{VIII.7}$$

That is, both the mean flow and the turbulent motion are themselves incompressible.

The total flow velocity $\vec{v}$ obeys the usual incompressible Navier–Stokes equation [cf. Eq. (III.33)]

$$\rho\left(\frac{\partial\vec{v}(t,\vec{r})}{\partial t} + \left[\vec{v}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{v}(t,\vec{r})\right) = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \eta\triangle\vec{v}(t,\vec{r}), \tag{VIII.8}$$

(lxiii) *Reynolds-Spannung*

from which the equations governing the mean and turbulent flows can be derived. For the sake of brevity, the arguments $(t,\vec{r})$ of the various fields will be omitted in the following.

VIII.1.3 a Equations for the mean flow

Inserting the Reynolds decompositions (VIII.2)–(VIII.3) into the Navier–Stokes equation (VIII.8) and averaging with the Reynolds average $\overline{\cdot}$ leads to the so-called *Reynolds equation*

$$\rho\left[\frac{\partial\overline{\vec{\mathsf{v}}}}{\partial t}+\left(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\right)\overline{\vec{\mathsf{v}}}\right]=-\vec{\nabla}\overline{\mathcal{P}}+\eta\triangle\overline{\vec{\mathsf{v}}}-\rho\overline{\left(\vec{\mathsf{v}}'\cdot\vec{\nabla}\right)\vec{\mathsf{v}}'}. \tag{VIII.9a}$$

To avoid confusion, this equation is also sometimes referred to as *Reynolds-averaged Navier–Stokes equation*. In terms of components in a given system of coordinates, this becomes, after dividing by the mass density ρ and accounting for incompressibility (see below)

$$\frac{\partial\overline{\mathsf{v}^i}}{\partial t}+\left(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\right)\overline{\mathsf{v}^i}=-\frac{1}{\rho}\frac{\mathrm{d}\overline{\mathcal{P}}}{\mathrm{d}x_i}-\sum_{j=1}^{3}\frac{\mathrm{d}\overline{\mathsf{v}'^i\mathsf{v}'^j}}{\mathrm{d}x^j}+\nu\triangle\overline{\mathsf{v}^i}\qquad\forall i=1,2,3. \tag{VIII.9b}$$

Using the incompressibility of the fluctuating motion, the rightmost term in Eq. (VIII.9a) can be rewritten as

$$-\rho\overline{\left(\vec{\mathsf{v}}'\cdot\vec{\nabla}\right)\vec{\mathsf{v}}'}=-\rho\vec{\nabla}\cdot\left(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'}\right)=-\vec{\nabla}\cdot\mathbf{T}_{\mathrm{R}}.$$

The Reynolds equation can thus be recast in the equivalent form [cf. Eq. (III.25b)]

$$\frac{\partial}{\partial t}\left(\rho\overline{\vec{\mathsf{v}}}\right)+\vec{\nabla}\cdot\overline{\mathbf{T}}=-\vec{\nabla}\cdot\mathbf{T}_{\mathrm{R}}, \tag{VIII.10}$$

with $\overline{\mathbf{T}}$ the momentum-flux density of the mean flow, given by [cf. Eqs. (III.27b), (III.27e)]

$$\overline{\mathbf{T}}\equiv\overline{\mathcal{P}}\,\mathbf{g}^{-1}+\rho\overline{\vec{\mathsf{v}}}\otimes\overline{\vec{\mathsf{v}}}-2\eta\overline{\mathbf{S}} \tag{VIII.11a}$$

i.e., in terms of components,

$$\overline{\mathbf{T}^{ij}}\equiv\overline{\mathcal{P}}g^{ij}+\rho\overline{\mathsf{v}^i}\,\overline{\mathsf{v}^j}-2\eta\overline{\mathbf{S}^{ij}}, \tag{VIII.11b}$$

where $\overline{\mathbf{S}}$ denotes the rate-of-shear tensor [Eq. (II.17b)] for the mean flow, whose components are given by [cf. Eq. (II.17d)]

$$\overline{\mathbf{S}^{ij}}\equiv\frac{1}{2}\left(\frac{\mathrm{d}\overline{\mathsf{v}^i}}{\mathrm{d}x_j}+\frac{\mathrm{d}\overline{\mathsf{v}^j}}{\mathrm{d}x_i}-\frac{2}{3}g^{ij}\vec{\nabla}\cdot\overline{\vec{\mathsf{v}}}\right). \tag{VIII.11c}$$

Note that the third term within the brackets actually vanishes due to the incompressibility of the mean flow, Eq. (VIII.7).

The form (VIII.10) of the Reynolds equation emphasizes perfectly the role of the Reynolds stress, i.e. the turbulent component of the flow, as "external" source driving the mean flow. In particular, the off-diagonal terms of the Reynolds stress describe sources of shear stresses, which will lead to the appearance of *eddies* in the flow.

Remark: The two equations (VIII.9a)–(VIII.9b) involve the material derivative "following the mean flow"

$$\frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}}t}\equiv\frac{\partial}{\partial t}+\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}, \tag{VIII.12}$$

which we shall further use in the remainder of this chapter.

Starting from the Reynolds equation (VIII.9), one can derive the equation governing the evolution of the kinetic energy $\frac{1}{2}\rho\left(\overline{\vec{\mathsf{v}}}\right)^2$ associated with the mean flow, namely

$$\frac{\overline{\mathrm{D}}}{\overline{\mathrm{D}}t}\left(\frac{\rho\overline{\vec{\mathsf{v}}}^2}{2}\right)=-\vec{\nabla}\cdot\left[\overline{\mathcal{P}}\overline{\vec{\mathsf{v}}}+\left(\mathbf{T}_{\mathrm{R}}-2\eta\overline{\mathbf{S}}\right)\cdot\overline{\vec{\mathsf{v}}}\right]+\left(\mathbf{T}_{\mathrm{R}}-2\eta\overline{\mathbf{S}}\right):\overline{\mathbf{S}}. \tag{VIII.13}$$

Traditionally, this equation is rather written in terms of the kinetic energy per unit mass $\bar{k} \equiv \frac{1}{2}(\vec{\overline{\mathsf{v}}})^2$, in which case it reads

$$\frac{\bar{\mathrm{D}}\bar{k}}{\bar{\mathrm{D}}t} = -\vec{\nabla}\cdot\left[\frac{1}{\rho}\overline{\mathcal{P}}\vec{\overline{\mathsf{v}}} + \left(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'} - 2\nu\overline{\mathbf{S}}\right)\cdot\vec{\overline{\mathsf{v}}}\right] + \left(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'} - 2\nu\overline{\mathbf{S}}\right):\overline{\mathbf{S}}, \tag{VIII.14a}$$

or in terms of components

$$\frac{\bar{\mathrm{D}}\bar{k}}{\bar{\mathrm{D}}t} = -\sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\overline{\mathsf{v}^j} + \sum_{i=1}^{3}\left(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\right)\overline{\mathsf{v}_i}\right] + \sum_{i,j=1}^{3}\left(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\right)\overline{\mathbf{S}_{ij}}. \tag{VIII.14b}$$

In either form, the physical meaning of each term is rather transparent: first comes the convective transport of energy in the mean flow, given by the divergence of the energy flux density, inclusive a term from the turbulent motion. The second term represents the energy which the mean flow "loses", namely either because it is dissipated by the viscous friction forces (term in $\nu\overline{\mathbf{S}}:\overline{\mathbf{S}}$), or because it is transferred to the turbulent part of the motion (term involving the Reynolds stress).

To prove Eq. (VIII.13), one should first average the inner product with $\vec{\overline{\mathsf{v}}}$ of the Reynolds equation (VIII.9), and then rewrite $\vec{\overline{\mathsf{v}}}\cdot\vec{\nabla}\overline{\mathcal{P}}$ and $\vec{\overline{\mathsf{v}}}\cdot(\vec{\mathsf{v}}'\cdot\vec{\nabla})\vec{\mathsf{v}}'$ under consideration of the incompressibility condition (VIII.7).

Remark: While equations (VIII.9) or (VIII.14) describe the dynamics of the mean flow, they rely on the Reynolds stress, which is not yet specified by the equations.

VIII.1.3 b Description of the transition to the turbulent regime

Turbulence takes place when the effects of Reynolds stress $\mathbf{T}_{\mathrm{R}}$—which represents a turbulent transport of momentum—predominates over those of the viscous stress tensor $2\rho\nu\overline{\mathbf{S}}$ associated with the mean flow, i.e. when the latter can no longer dampen the fluctuations corresponding to the former.

Let v_c resp. L_c denote a characteristic velocity resp. length scale of the fluid motion. Assuming that averages—here, a simple average over the volume is meant—over the flow yield the typical orders of magnitude

$$\left\langle\sum_{i,j=1}^{3}\left|\overline{\mathsf{v}'^i\mathsf{v}'^j}\overline{\mathbf{S}_{ij}}\right|\right\rangle \sim \frac{\mathsf{v}_c^3}{L_c} \quad\text{and}\quad \left\langle\sum_{i,j=1}^{3}\left|\nu\overline{\mathbf{S}^{ij}}\overline{\mathbf{S}_{ij}}\right|^2\right\rangle \sim \frac{\nu\mathsf{v}_c^2}{L_c^2}, \tag{VIII.15}$$

then in the turbulent regime the first of these terms is significantly larger than the second, which corresponds to having a large value of the Reynolds number $\mathrm{Re} \equiv \mathsf{v}_c L_c/\nu$ [Eq. (V.12)], in agreement with the qualitative discussion in § VIII.1.1 b.

In that situation, the equation (VIII.14) describing the evolution of the kinetic energy of the mean flow becomes

$$\frac{\bar{\mathrm{D}}\bar{k}}{\bar{\mathrm{D}}t} = -\vec{\nabla}\cdot\left[\frac{1}{\rho}\overline{\mathcal{P}}\vec{\overline{\mathsf{v}}} + \left(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'}\right)\cdot\vec{\overline{\mathsf{v}}}\right] + \left(\overline{\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'}\right):\overline{\mathbf{S}}, \tag{VIII.16a}$$

or component-wise

$$\frac{\bar{\mathrm{D}}\bar{k}}{\bar{\mathrm{D}}t} = -\sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\overline{\mathsf{v}^j} + \sum_{i=1}^{3}\left(\overline{\mathsf{v}'^i\mathsf{v}'^j}\right)\overline{\mathsf{v}_i}\right] + \sum_{i,j=1}^{3}\overline{\mathsf{v}'^i\mathsf{v}'^j}\overline{\mathbf{S}_{ij}}. \tag{VIII.16b}$$

That is, the viscosity is no longer a relevant parameter for the dynamics of the mean flow.

As already discussed above, the first term on the right hand side of Eq. (VIII.16) represents the convective transport of energy in the mean flow, while the second term describes the transfer of energy from the mean flow into the turbulent motion, and thus corresponds to the energy "dissipated" by the mean flow. Invoking the first relation in Eq. (VIII.15), the rate of energy dissipation per unit mass in the mean flow is thus

$$\dot{\mathcal{E}}_{\text{diss.}} = \left\langle \sum_{i,j=1}^{3} \overline{\mathsf{v}'^i \mathsf{v}'^j} \mathbf{S}_{ij} \right\rangle \sim \frac{\mathsf{v}_c^3}{L_c}. \tag{VIII.17}$$

This grows like the third power of the typical velocity, i.e. faster than v_c^2, as was mentioned at the end of § VIII.1.1 a for the turbulent regime of the Hagen–Poiseuille flow. In addition, this energy dissipation rate is actually independent of the properties (mass density, viscosity...) of the flowing fluid: turbulence is a characteristic of the motion, not of the fluid itself.

Eventually, the middle term in Eq. (VIII.17) must be negative, so that the energy really flows from the mean flow to the turbulent motion, not in the other direction!

Remark: Looking naively at the definition of the Reynolds number, the limit of an infinitely large Re corresponds to the case of a vanishing shear viscosity, that is, to the model of a perfect fluid. As was just discussed, this is clearly not the case: with growing Reynolds number, i.e. increasing influence of the turbulent motion, the number of eddies in the flow also increases, in which energy is dissipated into heat. In contrast, the kinetic energy is conserved in the flow of a perfect fluid. The solution to this apparent paradox is simply that with increasing Reynolds number, the velocity gradients in the flow also increase. In the incompressible Navier–Stokes equation, the growth of $\triangle\vec{\mathsf{v}}$ compensates the decrease of the viscosity ν, so that the corresponding term does not disappear and the Navier–Stokes equation does not simplify to the Euler equation.

VIII.1.4 Necessity of a statistical approach

As noted above, the evolution equation for the mean flow involves the Reynolds stress, for which no similar equation has yet be determined.

A first, natural solution is simply to write down the evolution equation for the turbulent velocity $\vec{\mathsf{v}}'(t,\vec{r})$, see Eq. (VIII.25) below. Invoking then the identity

$$\frac{\partial}{\partial t}\Big[\rho\,\overline{\vec{\mathsf{v}}'(t,\vec{r})\otimes\vec{\mathsf{v}}'(t,\vec{r})}\Big] = \rho\,\overline{\frac{\partial\vec{\mathsf{v}}'(t,\vec{r})}{\partial t}\otimes\vec{\mathsf{v}}'(t,\vec{r})} + \rho\,\overline{\vec{\mathsf{v}}'(t,\vec{r})\otimes\frac{\partial\vec{\mathsf{v}}'(t,\vec{r})}{\partial t}},$$

one can derive a dynamical equation for $\mathbf{T}_{\text{R}}$, the so-called *Reynolds-stress equation*[(lxiv)], which in component form reads

$$\frac{\overline{\mathrm{D}}\mathbf{T}_{\text{R}}^{ij}}{\overline{\mathrm{D}}t} = -2\overline{\mathcal{P}'\mathbf{S}'^{ij}} + \sum_{k=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^k}\left(\overline{\mathcal{P}'\mathsf{v}'^i}g^{jk} + \overline{\mathcal{P}'\mathsf{v}'^j}g^{ik} + \rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k} - \nu\frac{\mathrm{d}\mathbf{T}_{\text{R}}^{ij}}{\mathrm{d}x_k}\right)$$
$$- \sum_{k=1}^{3}\left(\mathbf{T}_{\text{R}}^{ik}\frac{\mathrm{d}\overline{\mathsf{v}}^j}{\mathrm{d}x^k} + \mathbf{T}_{\text{R}}^{jk}\frac{\mathrm{d}\overline{\mathsf{v}}^i}{\mathrm{d}x^k}\right) - 2\eta\sum_{k=1}^{3}\overline{\frac{\mathrm{d}\mathsf{v}'^i}{\mathrm{d}x^k}\frac{\mathrm{d}\mathsf{v}'^j}{\mathrm{d}x_k}}. \tag{VIII.18}$$

Irrespective of the physical interpretation of each of the terms in this equation, an important issue is that the evolution of $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j}$ involves a contribution from the components $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k}$ of a tensor of degree 3. In turn, the evolution of $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k}$ involves the tensor with components $\rho\overline{\mathsf{v}'^i\mathsf{v}'^j\mathsf{v}'^k\mathsf{v}'^l}$, and so on: at each step, the appearance of a tensor of higher degree simply reflects the nonlinearity of the Navier–Stokes equation.

All in all, the incompressible Navier–Stokes equation (VIII.8) is thus equivalent to an infinite hierarchy of equations relating the successive n-point autocorrelation functions of the fluctuations of the velocity field. Any subset of this hierarchy is not closed and involves more unknown fields than equations. A closure prescription, based on some physical assumption, is therefore necessary to obtain a description with a finite number of equations governing the (lower-order) autocorrelation functions. Such an approach is presented in Sec. VIII.2.

(lxiv) *Reynolds-Spannungsgleichung*

An alternative possibility is to assume directly some ansatz for the statistical behavior of the turbulent velocity, especially for its general two-point autocorrelation function, of which the equal-time and position correlator $\overline{\mathsf{v}'^i(t,\vec{r})\mathsf{v}'^j(t,\vec{r})}$ is only a special case. This avenue will be pursued in Sec. VIII.3.