VI.2 Shock waves

When the amplitude of the perturbations considered in Sec. VI.1 cannot be viewed as small, as for instance if $|\delta\vec{\mathsf{v}}| \ll c_s$ does not hold, the linearization of the equations of motion (VI.3) is no longer licit, and the nonlinear terms of the Euler equation play a role.

A possibility is then that at a finite time t in the evolution of the fluid, a discontinuity in some of the fields may appear, referred to as shock wave.(lxi) How this may arise will be discussed in the case of a one-dimensional problem (§ VI.2.1). At a discontinuity, the differential formulation of the conservation laws derived in Chap. III no longer holds, and it becomes necessary to study the conservation of mass, momentum and energy across the surface of discontinuity associated with the shock wave (§ VI.2.2).

VI.2.1 Formation of a shock wave in a one-dimensional flow

As in § VI.1.1, we consider the propagation of an adiabatic perturbation of a background fluid at rest, in the absence of gravity or of other external volume forces. In the one-dimensional case, the dynamical equations (VI.3) read

$$\frac{\partial\rho(t,x)}{\partial t} + \rho(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} + \delta\mathsf{v}(t,x)\frac{\partial\rho(t,x)}{\partial x} = 0, \tag{VI.23a}$$

$$\rho(t,x)\left[\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} + \delta\mathsf{v}(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}\right] + \frac{\partial\delta\mathcal{P}(t,x)}{\partial x} = 0. \tag{VI.23b}$$

The variation of the pressure $\delta\mathcal{P}(t,x)$ can again be expressed in terms of the variation in the mass density $\delta\rho(t,x)$ by invoking a Taylor expansion [cf. the paragraph between Eqs. (VI.4) and (VI.5)]. Since the perturbation of the background "flow" is no longer small, the thermodynamic state around which this Taylor expansion is performed is not necessarily that corresponding to the unperturbed fluid, but rather an arbitrary state, so that

$$\delta\mathcal{P}(t,x) \simeq c_s\big(\rho(t,x)\big)^2\delta\rho(t,x), \tag{VI.24}$$

where the speed of sound is that in the perturbed flow. When differentiating this identity, the derivative of $\delta\rho(t,x)$ with respect to x is also the derivative of $\rho(t,x)$, since the unperturbed fluid state is uniform. Accordingly, one may recast Eqs. (VI.23) as

$$\frac{\partial\rho(t,x)}{\partial t} + \rho(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} + \delta\mathsf{v}(t,x)\frac{\partial\rho(t,x)}{\partial x} = 0, \tag{VI.25a}$$

(lxi) *Stoßwelle*

$$\rho(t,x)\left[\frac{\partial\delta\mathsf{v}(t,x)}{\partial t}+\delta\mathsf{v}(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}\right]+c_s(\rho)^2\frac{\partial\rho(t,x)}{\partial x}=0, \tag{VI.25b}$$

which constitutes a system of two coupled partial differential equations for the two unknown fields $\rho(t,x)$ and $\delta\mathsf{v}(t,x)=\mathsf{v}(t,x)$.

To deal with these equations, one may assume that the mass density and the flow velocity have parallel dependences on time and space—as suggested by the fact that this property holds in the linearized case of sound waves, in which both $\rho(t,\vec{r})$ and $\vec{\mathsf{v}}(t,\vec{r})$ propagate with the same phase $(c_s|\vec{k}|t+\vec{k}\cdot\vec{r})$. Thus, the dependence of v on t and x is replaced with a functional dependence $\mathsf{v}\big(\rho(t,x)\big)$, with the known value $\mathsf{v}(\rho_0)=0$ corresponding to the unperturbed fluid at rest. Accordingly, the partial derivatives of the flow velocity with respect to t resp. x become

$$\frac{\partial\mathsf{v}(t,x)}{\partial t}=\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}\frac{\partial\rho(t,x)}{\partial t}\qquad\text{resp.}\qquad\frac{\partial\mathsf{v}(t,x)}{\partial x}=\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}\frac{\partial\rho(t,x)}{\partial x}.$$

The latter identities may then be inserted in Eqs. (VI.25). If one further multiplies Eq. (VI.25a) by $\rho(t,x)\,\mathrm{d}\mathsf{v}(\rho)/\mathrm{d}\rho$ and then subtracts Eq. (VI.25b) from the result, there comes

$$\left\{\rho^2\left[\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}\right]^2-c_s(\rho)^2\right\}\frac{\partial\rho(t,x)}{\partial x}=0,$$

that is, discarding the trivial solution of a uniform mass density,

$$\frac{\mathrm{d}\mathsf{v}(\rho)}{\mathrm{d}\rho}=\pm\frac{c_s(\rho)}{\rho}. \tag{VI.26}$$

The equations (VI.25)-(VI.26) are invariant under the simultaneous changes $\mathsf{v}\to-\mathsf{v}$, $x\to-x$, and $c_s\to-c_s$. Accordingly, one may restrict the discussion of Eq. (VI.26) to the case with a + sign—the − case amounts to considering a wave propagating in the opposite direction with the opposite velocity. The flow velocity is then formally given by

$$\mathsf{v}(\rho)=\int_{\rho_0}^{\rho}\frac{c_s(\rho')}{\rho'}\,\mathrm{d}\rho',$$

where we used $\mathsf{v}(\rho_0)=0$, while Eq. (VI.25b) can be rewritten as

$$\frac{\partial\rho(t,x)}{\partial t}+\left[\mathsf{v}\big(\rho(t,x)\big)+c_s\big(\rho(t,x)\big)\right]\frac{\partial\rho(t,x)}{\partial x}=0. \tag{VI.27}$$

Assuming that the mass density perturbation locally propagates as a traveling wave, i.e. making the ansatz[37] $\delta\rho(t,x)\propto f(x-c_wt)$ in Eq. (VI.27), then its phase velocity c_w will be given by $c_w=c_s(\rho)+\mathsf{v}$. Invoking Eq. (VI.26) then shows that $\mathrm{d}\mathsf{v}(\rho)/\mathrm{d}\rho>0$, so that c_w grows with increasing mass density: the denser regions in the fluid will propagate faster than the rarefied ones and possibly catch up with them—in case the latter where "in front" of the propagating perturbation—as illustrated in Fig. VI.1. In particular, there may arise after a finite amount of time a *discontinuity* of the function $\rho(t,x)$ at a given point x_0. The (propagating) point where this discontinuity takes place represents the front of a *shock wave*.

VI.2.2 Jump equations at a surface of discontinuity

To characterize the properties of a flow in the region of a shock wave, one needs first to specify the behavior of the physical quantities of relevance at the discontinuity, which is the object of this section. Generalizing the finding of the previous section in a one-dimensional setup, in which the discontinuity arises at a single (traveling) point, in the three-dimensional case there will be a whole *surface of discontinuity*,[lxii] that propagates in the unperturbed background fluid.

[37] This form is to be seen as the *local* form of the solution, not as a globally valid solution.

[lxii] *Unstetigkeitsfläche*

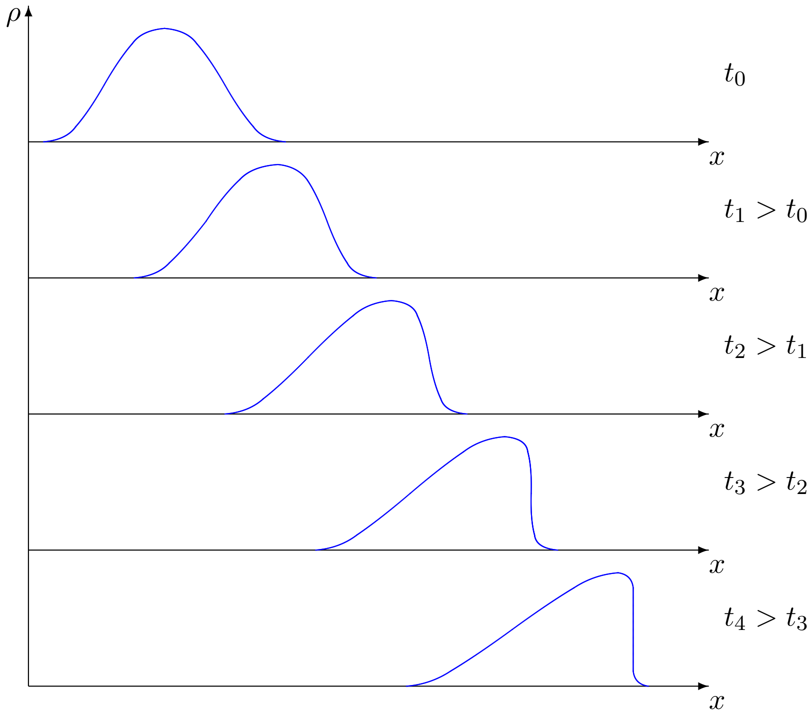

Figure VI.1 – Schematic representation of the evolution in time of the spatial distribution of dense and rarefied regions leading to a shock wave.

For the sake of brevity, the dependence on t and $\vec{r}$ of the various fields of interest will be omitted.

To describe the physics at the front of the shock wave, we adopt a comoving reference frame $\mathcal{R}$, which moves with the surface of discontinuity, and in this reference frame we consider a system of Cartesian coordinates (x^1, x^2, x^3) with the basis vector $\vec{e}_1$ perpendicular to the propagating surface. The region in front resp. behind the surface will be denoted by $(+)$ resp. $(-)$; that is, the fluid in which the shock waves propagates flows from the $(+)$- into the $(-)$-region: the former is upstream, the latter downstream.

The jump of a local physical quantity $g(\vec{r})$ across the surface of discontinuity is defined as

$$\llbracket g \rrbracket \equiv g_+ - g_-, \tag{VI.28}$$

where g_+ resp. g_- denotes the limiting value of g as $x^1 \to 0^+$ resp. $x^1 \to 0^-$. In case such a local quantity is actually continuous at the surface of discontinuity, then its jump across the surface vanishes.

At a surface of discontinuity $\mathcal{S}_{\rm d}$, the flux densities of mass, momentum, and energy across the surface, i.e. along the x^1-direction, must be continuous, so that mass, momentum, and energy remain locally conserved. These requirements are expressed by the *jump equations*[lxiii]

$$\llbracket \rho \mathsf{v}^1 \rrbracket = 0, \tag{VI.29a}$$

$$\llbracket \mathbf{T}^{i1} \rrbracket = 0 \quad \forall i = 1, 2, 3, \tag{VI.29b}$$

$$\left\llbracket \left(\frac{1}{2}\rho \vec{\mathsf{v}}^2 + e + \mathcal{P} \right) \mathsf{v}^1 \right\rrbracket = 0, \tag{VI.29c}$$

where the momentum flux density tensor has components $\mathbf{T}^{ij} = \mathcal{P} g^{ij} + \rho \mathsf{v}^i \mathsf{v}^j$ [see Eq. (III.24b)], with $g^{ij} = \delta^{ij}$ in the case of Cartesian coordinates.

[lxiii] *Sprunggleichungen*

The continuity of the mass flux density across the surface of discontinuity (VI.29a) can be recast as

$$(\rho \mathsf{v}^1)_- = (\rho \mathsf{v}^1)_+ \equiv j_1. \tag{VI.30}$$

A first, trivial solution arises if there is no flow of matter across the surface of discontinuity $\mathcal{S}_\mathrm{d}$, i.e. if $(\mathsf{v}^1)_+ = (\mathsf{v}^1)_- = 0$. In that case, Eq. (VI.29c) is automatically satisfied. Condition (VI.29b) for $i = 1$ becomes $[\![\mathcal{P}]\!] = 0$, i.e. the pressure is the same on both sides of $\mathcal{S}_\mathrm{d}$. Eventually, Eq. (VI.29b) with $i = 2$ or 3 holds automatically. All in all, there is no condition on the behavior of ρ, v^2 or v^3 across the surface of discontinuity—which means that these quantities may be continuous or not, in the latter case with an arbitrary jump.

If j_1 does not vanish, that is if some matter flows across $\mathcal{S}_\mathrm{d}$, then the jump equation for the component $\mathbf{T}^{21} = \rho \mathsf{v}^2 \mathsf{v}^1$ resp. $\mathbf{T}^{31} = \rho \mathsf{v}^3 \mathsf{v}^1$ leads to $[\![\mathsf{v}^2]\!] = 0$ resp. $[\![\mathsf{v}^3]\!] = 0$, i.e. the component v^2 resp. v^3 is continuous across the surface of discontinuity:

$$(\mathsf{v}^2)_- = (\mathsf{v}^2)_+ \quad \text{resp.} \quad (\mathsf{v}^3)_- = (\mathsf{v}^3)_+. \tag{VI.31}$$

In turn, rewriting the jump equation for $\mathbf{T}^{11} = \mathcal{P} + \rho(\mathsf{v}^1)^2$ with the help of j_1 yields

$$\mathcal{P}_- - \mathcal{P}_+ = j_1\big[(\mathsf{v}^1)_+ - (\mathsf{v}^1)_-\big] = j_1^2\left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right). \tag{VI.32}$$

Thus if $\rho_+ < \rho_-$, i.e. if the fluid is denser in the $(-)$-region downstream—as suggested by Fig. VI.1—, then $\mathcal{P}_- > \mathcal{P}_+$, while relation (VI.30) yields $(\mathsf{v}^1)_+ > (\mathsf{v}^1)_-$:[(38)]

$$\rho_- > \rho_+\,, \quad \mathcal{P}_- > \mathcal{P}_+\,, \quad (\mathsf{v}^1)_+ > (\mathsf{v}^1)_-. \tag{VI.33}$$

By invoking the necessary non-decrease of entropy, one can show (see Ref. [4], § 87) that these inequalities indeed hold.

Combining Eqs. (VI.30) and (VI.32) yields

$$\big[(\mathsf{v}^1)_+\big]^2 = \frac{j_1^2}{\rho_+^2} = \frac{\mathcal{P}_- - \mathcal{P}_+}{\rho_- - \rho_+}\frac{\rho_-\rho_+}{\rho_+^2} = \frac{\mathcal{P}_- - \mathcal{P}_+}{\rho_- - \rho_+}\frac{\rho_-}{\rho_+}$$

and similarly

$$\big[(\mathsf{v}^1)_-\big]^2 = \frac{\mathcal{P}_- - \mathcal{P}_+}{\rho_- - \rho_+}\frac{\rho_+}{\rho_-}.$$

If the jumps in pressure and mass density are small, one can show that their ratio is approximately the derivative $\partial\mathcal{P}/\partial\rho$, here at constant entropy and particle number, evaluated in the vicinity of the point where the jump occurs, i.e.

$$\big[(\mathsf{v}^1)_+\big]^2 \simeq \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}\frac{\rho_-}{\rho_+} = \frac{\rho_-}{\rho_+}c_s^2, \qquad \big[(\mathsf{v}^1)_-\big]^2 \simeq \frac{\rho_+}{\rho_-}c_s^2.$$

With $\rho_- > \rho_+$ comes $(\mathsf{v}^1)_+ > c_s$ resp. $(\mathsf{v}^1)_- < c_s$ upstream resp. downstream of the shock front.[(39)] The former identity means that an observer comoving with the surface of discontinuity sees in front of her/him a fluid flowing with a supersonic velocity, that is, going temporarily back to a reference frame bound to the unperturbed fluid, the shock wave moves with a supersonic velocity.

(38) Conversely, $\rho_+ > \rho_-$ would lead to $\mathcal{P}_- < \mathcal{P}_+$ and $(\mathsf{v}^1)_+ < (\mathsf{v}^1)_-$.

(39) Here we are being a little sloppy: one should consider the right ($x^1 \to 0^+$) and left ($x^1 \to 0^-$) derivatives, corresponding respectively to the $(+)$ and $(-)$-regions, and thus find the associated speeds of sound $(c_s)_+$ and $(c_s)_-$ instead of a single c_s.

Invoking the continuity across $\mathcal{S}_{\rm d}$ of the product $\rho\,\mathsf{v}^1$ and of the components v^2, v^3 parallel to the surface of discontinuity, the jump equation (VI.29c) for the energy flux density simplifies to

$$\left[\!\!\left[\frac{1}{2}(\mathsf{v}^1)^2+\frac{e+\mathcal{P}}{\rho}\right]\!\!\right]=\frac{j_1^2}{2}\left(\frac{1}{\rho_+^2}-\frac{1}{\rho_-^2}\right)+\frac{e_++\mathcal{P}_+}{\rho_+}-\frac{e_-+\mathcal{P}_-}{\rho_-}=0.$$

Expressing j_1^2 with the help of Eq. (VI.32), one finds

$$\frac{\mathcal{P}_--\mathcal{P}_+}{2}\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)=\frac{w_-}{\rho_-}-\frac{w_+}{\rho_+} \tag{VI.34a}$$

with $w=e+\mathcal{P}$ the enthalpy density, or equivalently

$$\frac{\mathcal{P}_-+\mathcal{P}_+}{2}\left(\frac{1}{\rho_+}-\frac{1}{\rho_-}\right)=\frac{e_+}{\rho_+}-\frac{e_-}{\rho_-}. \tag{VI.34b}$$

Either of these equations represents a relation between the thermodynamic quantities on both sides of the surface of discontinuity, and defines in the space of the thermodynamic states of the fluid a so-called *shock adiabatic curve*, also referred to as *dynamical adiabatic curve*[(lxiv)] or *Hugoniot*[(bf)] *adiabatic curve*, or *Rankine*[(bg)]*–Hugoniot relation*.

More generally, Eqs. (VI.30)–(VI.34) relate the dynamical fields on both sides of the surface of discontinuity associated with a shock wave, and constitute the practical realization of the continuity conditions encoded in the jump equations (VI.29).

[(lxiv)] *dynamische Adiabate*

[(bf)] P. H. Hugoniot, 1851–1887 [(bg)] W. J. M. Rankine, 1820–1872