VI.3 Gravity waves

In this section, we investigate waves that are "driven" by gravity, in the sense that the latter is the main force that acts to bring back the perturbed fluid to its unperturbed, "background" state. Such perturbations are generically referred to as *gravity waves*.[(lxi)]

A first example is that of small perturbations at the free surface of a liquid originally at rest—the "waves" of everyday language. In that case, some external source, as e.g. wind or an earthquake, leads to a local rise of the fluid above its equilibrium level: gravity then acts against this rise and tends to bring back the liquid to its equilibrium position. In case the elevation caused by the perturbation is small compared to the sea depth, as well as in comparison to the perturbation wavelength, one has linear sea surface waves (§ VI.3.1). Another interesting case arises in shallow water, for perturbations whose horizontal extent is much larger than their vertical size, in which case one may find so-called *solitary waves* (§ VI.3.2).

Throughout this section, the flows—comprised of a background fluid at rest and the traveling perturbation—are supposed to be two-dimensional, with the x-direction along the propagation direction and the z-direction along the vertical, oriented upwards so that the acceleration due to gravity is $\vec{g} = -g\,\vec{e}_z$. The origin $z = 0$ is taken at the bottom of the sea / ocean, which for the sake of simplicity is assumed to be flat.

VI.3.1 Linear sea surface waves

A surface wave is a perturbation of the altitude—with respect to the sea bottom—of the free surface of the sea. The latter is displaced by an amount $\delta h(t,x)$ from its equilibrium position h_0, corresponding to a fluid at rest with a horizontal free surface. These variations in the position of

(lxi) *Schwerewellen*

the free surface signal the motion of the sea water, i.e. a flow, with a corresponding flow velocity throughout the sea $\vec{\mathsf{v}}(t,x,z)$.

We shall model this motion as vorticity-free, which allows us to introduce a velocity potential $\varphi(t,x,z)$, and assume that the mass density ρ of the sea water remains constant and uniform, i.e. we neglect its compressibility. The sea is supposed to occupy an unbounded region of space, which is a valid assumption if one is far from any coast.

VI.3.1 a Equations of motion and boundary conditions

Under the above assumptions, the equations of motion read [cf. Eqs. (IV.31) and (IV.32)]

$$-\frac{\partial\varphi(t,x,z)}{\partial t} + \frac{\big[\vec{\nabla}\varphi(t,x,z)\big]^2}{2} + \frac{\mathcal{P}(t,x,z)}{\rho} + gz = \text{constant},^{(40)} \qquad \text{(VI.35a)}$$

where gz is the potential energy per unit mass of water, and

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]\varphi(t,x,z) = 0. \qquad \text{(VI.35b)}$$

To fully specify the problem, boundary conditions are still needed. As in the generic case for potential flow (Sec. IV.4), these will be Neumann boundary conditions, involving the derivative of the velocity potential.

- At the bottom of the sea, the water can have no vertical motion, corresponding to the usual impermeability condition; that is

$$\mathsf{v}_z(z\!=\!0) = -\frac{\partial\varphi}{\partial z}\bigg|_{z=0} = 0. \qquad \text{(VI.36a)}$$

- At the free surface of the sea, the vertical component v_z of the flow velocity equals the velocity of the surface, i.e. it equals the rate of change of the position of the (material!) surface:

$$-\frac{\partial\varphi(t,x,z)}{\partial z}\bigg|_{z=h_0+\delta h(t,x)} = \frac{\mathrm{D}\delta h(t,x)}{\mathrm{D}t}.$$

  Using $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathsf{v}_x\frac{\partial}{\partial x} = \frac{\partial}{\partial t} - \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial x}$, this gives

$$\left[\frac{\partial\varphi(t,x,z)}{\partial z} + \frac{\partial\delta h(t,x)}{\partial t} - \frac{\partial\delta h(t,x)}{\partial x}\frac{\partial\varphi(t,x,z)}{\partial x}\right]_{z=h_0+\delta h(t,x)} = 0. \qquad \text{(VI.36b)}$$

- At the free surface of the sea, the pressure on the water side—right below the surface—is directly related to that just above the surface. The latter is assumed to be constant and equal to some value $\mathcal{P}_0$, which represents for instance the atmospheric pressure "at sea level". As a first approximation—whose physical content will be discussed in the remark at the end of this paragraph—, the pressure is equal on both sides of the sea surface:

$$\mathcal{P}\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \mathcal{P}_0. \qquad \text{(VI.36c)}$$

  Expressing the pressure with the help of Eq. (VI.35a), this condition may be recast as

$$\left[-\frac{\partial\varphi(t,x,z)}{\partial t} + \frac{\big[\vec{\nabla}\varphi(t,x,z)\big]^2}{2}\right]_{z=h_0+\delta h(t,x)} + g\,\delta h(t,x) = -\frac{\mathcal{P}_0}{\rho} - gh_0 + \text{constant}, \qquad \text{(VI.36d)}$$

  where the whole right hand side of the equation represents a new constant.

(40) Here and in Eq. (VI.36d) and (VI.40b) below, the unspecified constant is in fact time dependent, yet this plays no role for the further calculations.

Hereafter we look for solutions consisting of a velocity potential $\varphi(t,x,z)$ and a surface profile $\delta h(t,x)$, as determined by Eqs. (VI.35) with conditions (VI.36).

Remark: The assumption of an identical pressure on both sides of an interface—either between two immiscible liquids, or between a liquid and a gas, as here—is generally *not* warranted, unless the interface happens to be flat. If there is the least curvature, the *surface tension* associated with the interface will lead to a larger pressure inside the concavity of the interface. Neglecting this effect—which we shall consider again in § VI.3.2—is valid only if the typical radius of curvature of the interface, which as we shall see below is the wavelength of the surface waves, is "large", especially with respect to the deformation scale δh.

VI.3.1 b Harmonic wave assumption

Since the domain on which the wave propagates is unbounded, a natural ansatz for the solution of the Laplace equation (VI.35b) is that of a harmonic wave

$$\varphi(t,x,z) = f(z)\cos(kx-\omega t) \tag{VI.37}$$

propagating in the x-direction with a depth-dependent amplitude $f(z)$. Inserting this form in the Laplace equation yields the linear ordinary differential equation

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} - k^2 f(z) = 0,$$

whose obvious solution is $f(z) = a_1\,\mathrm{e}^{kz} + a_2\,\mathrm{e}^{-kz}$, with a_1 and a_2 two real constants.

The boundary condition (VI.36a) at the sea bottom $z=0$ gives $a_1 = a_2$, i.e.

$$\varphi(t,x,z) = C\cosh(kz)\cos(kx-\omega t), \tag{VI.38}$$

with C a real constant.

To make further progress with the equations of the system, and in particular to determine the profile of the free surface, further assumptions are needed, so as to obtain simpler equations. We shall now present a first such simplification, leading to linear waves. In § VI.3.2, another simplification—of a more complicated starting point—will be considered, which gives rise to (analytically tractable) nonlinear waves.

VI.3.1 c Linear waves

As in the case of sound waves, we now assume that the perturbations are "small", so as to be able to linearize the equations of motion and those expressing boundary conditions. Thus, we shall assume that the quadratic term $(\vec{\nabla}\varphi)^2$ is much smaller than $|\partial\varphi/\partial t|$, and that the displacement δh of the free surface from its rest position is much smaller than the equilibrium sea depth h_0.

> To fix ideas the "swell waves" observed far from any coast on the Earth oceans or seas have a typical wavelength λ of about 100 m and an amplitude δh_0 of 10 m or less—the shorter the wavelength, the smaller the amplitude—, while the typical sea/ocean depth h_0 is 1–5 km.

The assumption $(\vec{\nabla}\varphi)^2 \ll |\partial\varphi/\partial t|$ can on the one hand be made in Eq. (VI.35a), leading to

$$-\frac{\partial\varphi(t,x,z)}{\partial t} + \frac{\mathcal{P}(t,x,z)}{\rho} + gz = \frac{\mathcal{P}_0}{\rho} + gh_0, \tag{VI.39}$$

in which the right member represents the zeroth order, while the left member also contains first order terms, which must cancel each other for the identity to hold. On the other hand, taking also into account the assumption $|\delta h(t,x)| \ll h_0$, the boundary conditions (VI.36b) and (VI.36d) at the free surface of the sea can be rewritten as

$$\left.\frac{\partial\varphi(t,x,z)}{\partial z}\right|_{z=h_0} + \frac{\partial\delta h(t,x)}{\partial t} = 0 \tag{VI.40a}$$

and

$$-\left.\frac{\partial\varphi(t,x,z)}{\partial t}\right|_{z=h_0} + g\,\delta h(t,x) = \text{constant}, \tag{VI.40b}$$

respectively. Together with the Laplace differential equation (VI.35b) and the boundary condition at the sea bottom (VI.36a), the two equations (VI.40) constitute the basis of the *Airy*[(an)] *linear wave theory*.

Combining the latter two equations yields at once the condition

$$\left[\frac{\partial^2\varphi(t,x,z)}{\partial t^2} + g\frac{\partial\varphi(t,x,z)}{\partial z}\right]_{z=h_0} = 0.$$

Using the velocity potential (VI.38), this relation reads

$$-\omega^2 C\cosh(kh_0)\cos(kx-\omega t) + gkC\sinh(kh_0)\cos(kx-\omega t) = 0,$$

resulting in the *dispersion relation*

$$\omega^2 = gk\tanh(kh_0). \tag{VI.41}$$

This relation becomes even simpler in two limiting cases:

- When $kh_0 \gg 1$, or equivalently $h_0 \gg \lambda$ where $\lambda = 2\pi/k$ denotes the wavelength, which represents the case of gravity waves at the surface of deep sea,[(41)] then $\tanh(kh_0) \simeq 1$. In that case, the dispersion relation simplifies to $\omega^2 = gk$: the phase and group velocity of the traveling waves are

$$c_\varphi = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \quad\text{and}\quad c_g = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}} \tag{VI.42}$$

  respectively. Both are independent from the sea depth h_0, yet do depend on the angular wavenumber k of the wave, so that waves with different wavelengths propagate with different velocities.

- For $kh_0 \ll 1$, i.e. in the case of a shallow sea with $h_0 \ll \lambda$, the approximation $\tanh(kh_0) \simeq kh_0$ leads to the dispersion relation $\omega^2 = gh_0k^2$, i.e. to phase and group velocities

$$c_\varphi = c_g = \sqrt{gh_0}, \tag{VI.43}$$

  independent from the wavelength λ, signaling the absence of dispersive behavior.

  This phase velocity decreases with decreasing water depth h_0. Accordingly, this might lead to an accumulation, similar to the case of a shock wave in Sec. VI.2, whose description however requires that one take into account the nonlinear terms in the equations, which have been discarded here. In particular, we have explicitly assumed $|\delta h(t,x)| \ll h_0$, in order to linearize the problem, so that considering the limiting case $h_0 \to 0$ is questionable.

  In addition, a temptation when investigating the small-depth behavior $h_0 \to 0$ is clearly to describe the breaking of waves as they come to shore. Yet the harmonic ansatz (VI.38) assumes that the Laplace equation is considered on a horizontally unbounded domain, i.e. far from any coast, so again the dispersion relation (VI.41) may actually no longer be valid.

[(41)] The sea may not be "too deep", otherwise the assumed uniformity of the water mass density along the vertical direction in the unperturbed state does not hold. With $\lambda \simeq 100$ m, the inverse wave number is $k^{-1} \simeq 15$ m, so that $h_0 = 100$ m already represents a deep ocean; in comparison, the typical scale on which non-uniformities in the mass density are relevant is rather 1 km.

[(an)] G. B. Airy, 1801–1892

The boundary condition (VI.40b) provides us directly with the shape of the free surface of the sea, namely

$$\delta h(t,x) = \frac{1}{g}\frac{\partial\varphi(t,x,z)}{\partial t}\bigg|_{z=h_0} = \frac{\omega C}{g}\cosh(kh_0)\sin(kx-\omega t) \equiv \delta h_0\sin(kx-\omega t),$$

with $\delta h_0 \equiv (\omega C/g)\cosh(kh_0)$ the amplitude of the wave, which must remain much smaller than h_0. The profile of the surface waves of Airy's linear theory—or rather its cross section—is thus a simple sinusoidal curve.

This shape automatically suggests a generalization, which is a first step towards taking into account nonlinearities, such that the free surface profile is sum of (a few) harmonics $\sin(kx-\omega t)$, $\sin 2(kx-\omega t)$, $\sin 3(kx-\omega t)\ldots$ The approach leading to such a systematically expanded profile, which relies on a perturbative expansion to deal with the (still small) nonlinearities, is that of the *Stokes waves*.

The gradient of the potential (VI.38) yields (the components of) the flow velocity

$$\mathsf{v}_x(t,x,z) = \frac{kg}{\omega}\frac{\cosh(kz)}{\cosh(kh_0)}\delta h_0\sin(kx-\omega t),$$
$$\mathsf{v}_z(t,x,z) = -\frac{kg}{\omega}\frac{\sinh(kz)}{\cosh(kh_0)}\delta h_0\cos(kx-\omega t).$$

Integrating these functions with respect to time leads to the two functions

$$\mathsf{x}(t) = \mathsf{x}_0 + \frac{kg\delta h_0}{\omega^2}\frac{\cosh(kz)}{\cosh(kh_0)}\cos(kx-\omega t) = \mathsf{x}_0 + \frac{\delta h_0\cosh(kz)}{\sinh(kh_0)}\cos(kx-\omega t),$$
$$\mathsf{z}(t) = \mathsf{z}_0 + \frac{kg\delta h_0}{\omega^2}\frac{\sinh(kz)}{\cosh(kh_0)}\sin(kx-\omega t) = \mathsf{z}_0 + \frac{\delta h_0\sinh(kz)}{\sinh(kh_0)}\sin(kx-\omega t),$$

with x_0 and z_0 two integration constants. Choosing $\mathsf{x}_0 \simeq x$ and $\mathsf{z}_0 \simeq z$, if $\delta h_0 \ll k^{-1}$, these functions represent the components of the trajectory (pathline) of a fluid particle that is at time t in the vicinity of the point with coordinates (x,z), and whose velocity at that time is thus approximately the flow velocity $\vec{\mathsf{v}}(t,x,z)$. Since

$$\frac{[\mathsf{x}(t)-\mathsf{x}_0]^2}{\cosh^2(kz)} + \frac{[\mathsf{z}(t)-\mathsf{z}_0]^2}{\sinh^2(kz)} = \left[\frac{kg\delta h_0}{\omega^2\cosh(kh_0)}\right]^2 = \left[\frac{\delta h_0}{\sinh(kh_0)}\right]^2,$$

this trajectory is an ellipse, whose major and minor axes decrease with increasing depth $h_0 - z$. In the deep sea case $kh_0 \gg 1$, one can use the approximations $\sinh(kz) \simeq \cosh(kz) \simeq \mathrm{e}^{kz}/2$ for $1 \ll kz \lesssim kh_0$, which shows that the pathlines close to the sea surface are approximately circles.

Eventually, the pressure distribution in the sea follows from Eq. (VI.39) in which one uses the velocity potential (VI.38), resulting in

$$\mathcal{P}(t,x,z) = \mathcal{P}_0 + \rho g(h_0-z) + \rho\frac{\partial\varphi(t,x,z)}{\partial t} = \mathcal{P}_0 + \rho g\left[h_0 - z + \delta h_0\frac{\cosh(kz)}{\cosh(kh_0)}\sin(kx-\omega t)\right].$$

The contribution $\mathcal{P}_0 + \rho g(h_0-z)$ is the usual hydrostatic one, corresponding to the unperturbed sea, while the effect of the surface wave is proportional to its amplitude δh_0 and decreases with increasing depth.