Chapter VI

Waves in non-relativistic fluids

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A large class of solutions of the equations of motion (III.9), (III.18) and (III.35)—in the case of a perfect fluid—, or (III.9), (III.32), (III.37)—for a Newtonian fluid—, is that of waves. Quite generically, this denomination designates "perturbations" of some "unperturbed" fluid motion, which will also be referred to as "background flow".

In more mathematical terms, the starting point is a set of fields $\{\rho_0(t,\vec{r}), \vec{\mathsf{v}}_0(t,\vec{r}), \mathcal{P}_0(t,\vec{r})\}$ solving the equations of motion, that represent the background flow, either ideal or dissipative. The wave itself consists of a second set of fields $\{\delta\rho(t,\vec{r}), \delta\vec{\mathsf{v}}(t,\vec{r}), \delta\mathcal{P}(t,\vec{r})\}$, which are added on top of the background ones, such that the resulting fields

$$\rho(t,\vec{r}) = \rho_0(t,\vec{r}) + \delta\rho(t,\vec{r}), \tag{VI.1a}$$

$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0(t,\vec{r}) + \delta\mathcal{P}(t,\vec{r}), \tag{VI.1b}$$

$$\vec{\mathsf{v}}(t,\vec{r}) = \vec{\mathsf{v}}_0(t,\vec{r}) + \delta\vec{\mathsf{v}}(t,\vec{r}) \tag{VI.1c}$$

are again solutions to the same equations of motion.

Different kinds of perturbations—triggered by some source which will not be specified hereafter, and is thus to be seen as an initial condition—can be considered, leading to different phenomena.

A first distinction, with which the reader is probably already familiar, is that between traveling waves, which propagate, and standing waves, which do not. Mathematically, in the former case the propagating quantity does not depend on space and time independently, but rather on a combination like (in a one-dimensional case) $x - c_\varphi t$, where c_φ denotes some propagation speed. In contrast, in standing waves the space and time dependences of the "propagating" quantity factorize. Hereafter, we shall mostly mention traveling waves.

Another difference is that between "small" and "large" perturbations or, in more technical terms, between linear and nonlinear waves. In the former case, which is that of sound waves (Sec. VI.1) or the simplest gravity-controlled surface waves in liquids (§ VI.3.1), the partial differential equation governing the propagation of the wave is linear—which means that nonlinear terms have been neglected. Quite obviously, nonlinearities of the dynamical equations—as e.g. the Euler equation—are the main feature of nonlinear waves, as for instance shock waves (VI.2) or solitons (§ VI.3.2).

VI.1 Sound waves

By definition, the phenomenon which in everyday life is referred to as "sound" consists of *small adiabatic pressure perturbations* around a background flow, where adiabatic actually means that the entropy remains constant. In the presence of such a wave, each point in the fluid undergoes alternative *compression* and *rarefaction* processes. That is, these waves are by construction (parts of) a compressible flow.

We shall first consider sound waves on a uniform perfect fluid at rest (§ VI.1.1).
What then? Doppler effect? Rarefaction waves? Eventually, we discuss how viscous effects in a Newtonian fluid lead to the absorption of sound waves (§ VI.1.4).

VI.1.1 Sound waves in a uniform fluid at rest

Assuming that there are no external forces, a trivial solution of the dynamical equations of perfect fluids is that with uniform and time independent mass density ρ_0 and pressure $\mathcal{P}_0$, with a vanishing flow velocity $\vec{\mathrm{v}}_0 = \vec{0}$. Assuming in addition that the particle number N_0 in the fluid is conserved, its total entropy has a fixed value S_0. These conditions will represent the background flow we consider hereafter.

With the various fields that were just specified, a perturbation (VI.1) of this background flow reads

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}), \tag{VI.2a}$$

$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \delta\mathcal{P}(t,\vec{r}), \tag{VI.2b}$$

$$\vec{\mathrm{v}}(t,\vec{r}) = \vec{0} + \delta\vec{\mathrm{v}}(t,\vec{r}). \tag{VI.2c}$$

The necessary "smallness" of perturbations means for the mass density and pressure terms

$$|\delta\rho(t,\vec{r})| \ll \rho_0, \quad |\delta\mathcal{P}(t,\vec{r})| \ll \mathcal{P}_0. \tag{VI.2d}$$

Regarding the velocity, the background flow does not explicitly specify a reference scale, with which the perturbation should be compared. As we shall see below, the reference scale is actually implicitly contained in the equation(s) of state of the fluid under consideration, and the condition of small perturbation reads

$$|\delta\vec{\mathrm{v}}(t,\vec{r})| \ll c_s \tag{VI.2e}$$

with c_s the *speed of sound* in the fluid.

Inserting the fields (VI.2) in the equations of motion (III.9) and (III.18) and taking into account the uniformity and stationarity of the background flow, one finds

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r}) + \vec{\nabla}\cdot\big[\delta\rho(t,\vec{r})\,\delta\vec{\mathrm{v}}(t,\vec{r})\big] = 0, \tag{VI.3a}$$

$$\big[\rho_0 + \delta\rho(t,\vec{r})\big]\left\{\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\delta\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\delta\vec{\mathrm{v}}(t,\vec{r})\right\} + \vec{\nabla}\delta\mathcal{P}(t,\vec{r}) = \vec{0}. \tag{VI.3b}$$

The required smallness of the perturbations will help us simplify these equations, in that we shall only keep the leading-order terms in an expansion in which we consider ρ_0, $\mathcal{P}_0$ as zeroth-order quantities while $\delta\rho(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ and $\delta\vec{\mathrm{v}}(t,\vec{r})$ are small quantities of first order. Accordingly, the third term in the continuity equation is presumably much smaller than the other two, and may be left aside in a first approximation. Similarly, the contribution of $\delta\rho(t,\vec{r})$ and the convective term within the curly brackets on the left hand side of Eq. (VI.3b) may be dropped. The equations describing the coupled evolutions of $\delta\rho(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ and $\delta\vec{\mathrm{v}}(t,\vec{r})$ are thus *linearized*:

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r}) = 0, \tag{VI.4a}$$

$$\rho_0\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \vec{\nabla}\delta\mathcal{P}(t,\vec{r}) = \vec{0}. \tag{VI.4b}$$

To obtain a closed system of equations, a further relation between the perturbations is needed. This will be provided by thermodynamics, i.e. by the implicit assumption that the fluid at rest is everywhere in a state in which its pressure $\mathcal{P}$ is function of mass density ρ, (local) entropy S, and (local) particle number N, i.e. that there exists a unique relation $\mathcal{P} = \mathcal{P}(\rho, S, N)$ which is valid at each point in the fluid and at every time. Expanding this relation around the (thermodynamic) point corresponding to the background flow, namely $\mathcal{P}_0 = \mathcal{P}(\rho_0, S_0, N_0)$, one may write

$$\mathcal{P}\big(\rho_0 + \delta\rho, S_0 + \delta S, N_0 + \delta N\big) = \mathcal{P}_0 + \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}\delta\rho + \left(\frac{\partial\mathcal{P}}{\partial S}\right)_{\rho,N}\delta S + \left(\frac{\partial\mathcal{P}}{\partial N}\right)_{S,\rho}\delta N,$$

where the derivatives are to be evaluated at the point (ρ_0, S_0, N_0). Here, we wish to consider isentropic perturbations at constant particle number, i.e. both δS and δN vanish, leaving

$$\delta\mathcal{P} = \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}\delta\rho.$$

For the partial derivative of the pressure, we introduce the notation

$$c_s^2 \equiv \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N} \tag{VI.5}$$

where both sides actually depend on ρ_0, S_0 and N_0, yielding

$$\delta\mathcal{P} = c_s^2\,\delta\rho.$$

This thermodynamic relation holds at each point of the fluid at each instant, so that one can now replace $\vec{\nabla}\delta\mathcal{P}(t,\vec{r})$ by $c_s^2\,\vec{\nabla}\delta\rho(t,\vec{r})$ in Eq. (VI.4b):

$$\rho_0\frac{\partial\delta\vec{v}(t,\vec{r})}{\partial t} + c_s^2\,\vec{\nabla}\delta\rho(t,\vec{r}) = \vec{0}. \tag{VI.4c}$$

The equations (VI.4a), (VI.4c) for the perturbations $\delta\rho(t,\vec{r})$ and $\delta\vec{v}(t,\vec{r})$ are linear first order partial differential equations. Thanks to the linearity, their solutions form a vector space—at least as long as no initial condition has been specified. One can for instance express the solutions as Fourier transforms, i.e. superpositions of plane waves, characterized by their (angular) frequency ω and their wave vector $\vec{k}$. Accordingly, we test the ansatz

$$\delta\rho(t,\vec{r}) = \widetilde{\delta\rho}(\omega,\vec{k})\,\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\vec{k}\cdot\vec{r}}, \qquad \delta\vec{v}(t,\vec{r}) = \widetilde{\delta\vec{v}}(\omega,\vec{k})\,\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\vec{k}\cdot\vec{r}}, \tag{VI.6}$$

with respective amplitudes $\widetilde{\delta\rho}$, $\widetilde{\delta\vec{v}}$ that a priori depend on ω and $\vec{k}$ and are determined by the initial conditions for the problem. In turn, ω and $\vec{k}$ are not necessarily independent from each other, as we shall indeed find hereafter.

With this ansatz, Eqs. (VI.4) become

$$-\mathrm{i}\omega\widetilde{\delta\rho}(\omega,\vec{k}) + \mathrm{i}\rho_0\,\vec{k}\cdot\widetilde{\delta\vec{v}}(\omega,\vec{k}) = 0 \tag{VI.7a}$$

$$-\mathrm{i}\omega\rho_0\,\widetilde{\delta\vec{v}}(\omega,\vec{k}) + \mathrm{i}c_s^2\,\vec{k}\,\widetilde{\delta\rho}(\omega,\vec{k}) = \vec{0}. \tag{VI.7b}$$

From the second equation, the amplitude $\widetilde{\delta\vec{v}}(\omega,\vec{k})$ is proportional to $\vec{k}$; in particular, it lies along the same direction. That is, the inner product $\vec{k}\cdot\widetilde{\delta\vec{v}}$ simply equals the product of the norms of the two vectors.

Omitting from now on the $(\omega,\vec{k})$-dependence of the amplitudes, the inner product of Eq. (VI.7b) with $\vec{k}$—which does not lead to any loss of information—allows one to recast the system as

$$\begin{pmatrix} -\omega & \rho_0 \\ c_s^2\vec{k}^2 & -\omega\rho_0 \end{pmatrix}\begin{pmatrix} \widetilde{\delta\rho} \\ \vec{k}\cdot\widetilde{\delta\vec{v}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A first, trivial solution to this system is $\widetilde{\delta\rho} = 0$, $\widetilde{\delta\vec{v}} = \vec{0}$, i.e. the absence of any perturbation. In order for non-trivial solutions to exist, the determinant $(\omega^2 - c_s^2\vec{k}^2)\rho_0$ of the system should vanish.

This leads at once to the *dispersion relation*

$$\omega = \pm c_s|\vec{k}|. \tag{VI.8}$$

Denoting by $\vec{e}_{\vec{k}}$ the unit vector in the direction of $\vec{k}$, the perturbations $\delta\rho(t,\vec{r})$ and $\delta\vec{v}(t,\vec{r})$ defined by Eq. (VI.6), as well as $\delta\mathcal{P}(t,\vec{r}) = c_s^2\,\delta\rho(t,\vec{r})$, are all functions of $c_st\pm\vec{r}\cdot\vec{e}_{\vec{k}}$. These are thus *traveling waves*,[(lv)] that propagate with the phase velocity $\omega(\vec{k})/|\vec{k}| = c_s$, which is independent of $\vec{k}$. That is, c_s is the *speed of sound*, and the latter is the same for all frequencies.
For instance, for air at $T = 300$ K, the speed of sound is $c_s = 347\ \mathrm{m\cdot s^{-1}}$.

Air is a diatomic ideal gas, i.e. it has pressure $\mathcal{P} = Nk_BT/\mathcal{V}$ and internal energy $U = \frac{5}{2}Nk_BT$, giving

$$c_s^2 = \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N} = -\frac{\mathcal{V}^2}{mN}\left(\frac{\partial\mathcal{P}}{\partial\mathcal{V}}\right)_{S,N} = -\frac{\mathcal{V}^2}{mN}\left[-\frac{Nk_BT}{\mathcal{V}^2} + \frac{Nk_B}{\mathcal{V}}\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N}\right].$$

The thermodynamic relation $\mathrm{d}U = T\,\mathrm{d}S - \mathcal{P}\,\mathrm{d}\mathcal{V} + \mu\,\mathrm{d}N$ yields at constant entropy and particle number

$$\mathcal{P} = -\left(\frac{\partial U}{\partial\mathcal{V}}\right)_{S,N} = -\frac{5}{2}Nk_B\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N} \qquad \text{i.e.} \qquad Nk_B\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N} = -\frac{2\mathcal{P}}{5} = -\frac{2}{5}\frac{Nk_BT}{\mathcal{V}}.$$

leading to $c_s^2 = \dfrac{7}{5}\dfrac{k_BT}{m_{\rm air}}$, with $m_{\rm air} = 29/\mathcal{N}_A\ \mathrm{g\cdot mol^{-1}}$. □

Remarks:

∗ Instead of c_s^2, one may use the fluid *isentropic compressibility*, defined as

$$\beta_S \equiv -\frac{1}{\mathcal{V}}\left(\frac{\partial\mathcal{V}}{\partial\mathcal{P}}\right)_S = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial\mathcal{P}}\right)_S, \tag{VI.9a}$$

to relate the variations of pressure and mass density $\delta\mathcal{P}$, $\delta\rho$. This compressibility is related to c_s (evaluated at $\rho = \rho_0$) by

$$\beta_S = \frac{1}{\rho_0 c_s^2} \quad \text{resp.} \quad c_s = \frac{1}{\sqrt{\rho_0\beta_S}}, \tag{VI.9b}$$

which shows that the sound velocity is larger in "stiffer" fluids, i.e. fluids with a smaller compressibility — as generally liquids with respect to gases.

∗ Taking the real parts of the complex quantities in the harmonic waves (VI.6), so as to obtain real-valued $\delta\rho$, $\delta\mathcal{P}$ and $\delta\vec{v}$, one sees that these will be alternatively positive and negative, and in average—over a duration much longer than a period $2\pi/\omega$—zero. This in particular means that the successive compression and condensation ($\delta\mathcal{P} > 0$, $\delta\rho > 0$) or depression and rarefaction[(lvi)] ($\delta\mathcal{P} < 0$, $\delta\rho < 0$) processes do not lead to a resulting transport of matter.

∗ A single harmonic wave (VI.6) is a traveling wave. Yet if the governing equation or systems of equations is linear or has been linearized, as was done here, the superposition of harmonic waves is a valid solution. In particular, the superposition of two harmonic traveling waves with equal frequencies ω, opposite wave vectors $\vec{k}$—which is allowed by the dispersion relation (VI.8)—and equal amplitudes leads to a *standing wave*, in which the dependence on time and space is proportional to $\mathrm{e}^{\mathrm{i}\omega t}\cos(\vec{k}\cdot\vec{r})$.

Coming back to Eq. (VI.7b), the proportionality of $\widetilde{\delta\vec{v}}(\omega,\vec{k})$ and $\vec{k}$ means that the sound waves in a fluid are *longitudinal*—in contrast to electromagnetic waves in vacuum, which are transversal waves.

The nonexistence of transversal waves in fluids reflects the absence of forces that would act against shear deformations so as to restore some equilibrium shape—shear viscous effects cannot play that role.

(lv) *fortschreitende Wellen* (lvi) *Verdünnung*

In contrast, there can be transversal sound waves in elastic solids, as e.g. the so-called S-modes (shear modes) in geophysics.

The inner product of Eq. (VI.7b) with $\vec{k}$, together with the dispersion relation (VI.8) and the collinearity of $\widetilde{\delta\vec{v}}$ and $\vec{k}$, leads to the relation

$$\omega\rho_0|\vec{k}||\widetilde{\delta\vec{v}}| = c_s^2\,|\vec{k}|\widetilde{\delta\rho} \qquad \Leftrightarrow \qquad \frac{|\widetilde{\delta\vec{v}}|}{c_s} = \frac{\widetilde{\delta\rho}}{\rho_0}$$

for the amplitudes of the perturbations. This justifies condition (VI.2e), which is then consistent with (VI.2d). Similarly, inserting the ansatz (VI.6) in Eq. (VI.3b), the terms within curly brackets become $-\mathrm{i}\omega\,\widetilde{\delta\vec{v}} + \mathrm{i}(\vec{k}\cdot\widetilde{\delta\vec{v}})\widetilde{\delta\vec{v}}$: again, neglecting the second with respect to the first is equivalent to requesting $|\widetilde{\delta\vec{v}}| \ll c_s$.

Remark: Going back to Eqs. (VI.4a) and (VI.4c), the difference of the time derivative of the first one and the divergence of the second one leads to the known *wave equation*[35]

$$\frac{\partial^2\delta\rho(t,\vec{r})}{\partial t^2} - c_s^2\triangle\delta\rho(t,\vec{r}) = 0, \tag{VI.10a}$$

If the flow—including the background flow on which the sound wave develops, in case $\vec{v}_0$ is not trivial as it was assumed here—is irrotational, so that one may write $\vec{v}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r})$, then the velocity potential φ also obeys the same equation

$$\frac{\partial^2\varphi(t,\vec{r})}{\partial t^2} - c_s^2\triangle\varphi(t,\vec{r}) = 0.$$

VI.1.2 Sound waves in a moving fluid

Doppler effect!

VI.1.3 Riemann problem. Rarefaction waves

Should be added at some point

VI.1.4 Absorption of sound waves

In chapter V, we only considered incompressible motions of Newtonian fluids, so that bulk viscosity could from the start play no role. The simplest example of compressible flow is that of sound waves. As in § VI.1.1, we consider small adiabatic perturbations of a fluid initially at rest and with uniform properties—which implies that external volume forces like gravity are neglected. Accordingly, the dynamical fields characterizing the fluid are

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}), \quad \mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \delta\mathcal{P}(t,\vec{r}), \quad \vec{v}(t,\vec{r}) = \vec{0} + \delta\vec{v}(t,\vec{r}), \tag{VI.11a}$$

with

$$|\delta\rho(t,\vec{r})| \ll \rho_0, \quad |\delta\mathcal{P}(t,\vec{r})| \ll \mathcal{P}_0, \quad |\delta\vec{v}(t,\vec{r})| \ll c_s, \tag{VI.11b}$$

where c_s denotes the quantity which in the perfect-fluid case was found to coincide with the phase velocity of similar small perturbations, i.e. the "speed of sound", defined by Eq. (VI.5)

$$c_s^2 \equiv \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}. \tag{VI.11c}$$

As in § VI.1.1, this partial derivative will allow us to relate the pressure perturbation $\delta\mathcal{P}$ to the variation of mass density $\delta\rho$.

[35] This traditional denomination is totally out of place in a chapter in which there are several types of waves, each of which has its own governing "wave equation". Yet historically, due to its role for electromagnetic or sound waves, it is *the* archetypal wave equation, while the equations governing other types of waves often have a specific name.

Remark: Anticipating on later findings, the perturbations must actually fulfill a further condition, related to the size of their spatial variations [cf. Eq. (VI.21)]. This is nothing but the assumption of "small gradients" that underlies the description of their propagation with the Navier–Stokes equation, i.e. with first-order dissipative fluid dynamics.

For the sake of simplicity, we consider a one-dimensional problem, i.e. perturbations propagating along the x-direction and independent of y and z—as are the properties of the underlying background fluid. Under this assumption, the continuity equation (III.9) reads

$$\frac{\partial\rho(t,x)}{\partial t} + \rho(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} + \delta\mathsf{v}(t,x)\frac{\partial\rho(t,x)}{\partial x} = 0, \tag{VI.12a}$$

while the Navier–Stokes equation (III.32) becomes

$$\rho(t,x)\left[\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} + \delta\mathsf{v}(t,x)\frac{\partial\delta\mathsf{v}(t,x)}{\partial x}\right] = -\frac{\partial\delta\mathcal{P}(t,x)}{\partial x} + \left(\frac{4}{3}\eta+\zeta\right)\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.12b}$$

Substituting the fields (VI.11a) in these equations and linearizing the resulting equations so as to keep only the leading order in the small perturbations, one finds

$$\frac{\partial\delta\rho(t,x)}{\partial t} + \rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial x} = 0, \tag{VI.13a}$$

$$\rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} = -\frac{\partial\delta\mathcal{P}(t,x)}{\partial x} + \left(\frac{4}{3}\eta+\zeta\right)\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.13b}$$

In the second equation, the derivative $\partial(\delta\mathcal{P})/\partial x$ can be replaced by $c_s^2\,\partial(\delta\rho)/\partial x$. Let us in addition introduce the (traditional) notation[36]

$$\bar\nu \equiv \frac{1}{\rho_0}\left(\frac{4}{3}\eta+\zeta\right), \tag{VI.14}$$

so that Eq. (VI.13b) can be rewritten as

$$\rho_0\frac{\partial\delta\mathsf{v}(t,x)}{\partial t} + c_s^2\frac{\partial\delta\rho(t,x)}{\partial x} = \rho_0\bar\nu\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2}. \tag{VI.15}$$

Subtracting c_s^2 times the derivative of Eq. (VI.13a) with respect to x from the time derivative of Eq. (VI.15) and dividing the result by ρ_0 then yields

$$\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial t^2} - c_s^2\frac{\partial^2\delta\mathsf{v}(t,x)}{\partial x^2} = \bar\nu\frac{\partial^3\delta\mathsf{v}(t,x)}{\partial t\,\partial x^2}. \tag{VI.16a}$$

One easily checks that the mass density variation obeys a similar equation

$$\boxed{\frac{\partial^2\delta\rho(t,x)}{\partial t^2} - c_s^2\frac{\partial^2\delta\rho(t,x)}{\partial x^2} = \bar\nu\frac{\partial^3\delta\rho(t,x)}{\partial t\,\partial x^2}.} \tag{VI.16b}$$

In the perfect-fluid case $\bar\nu = 0$, one recovers the traditional wave equation (VI.10a).

Equations (VI.16) are homogeneous linear partial differential equations, whose solutions can be written as superposition of plane waves. Accordingly, let us substitute the Fourier ansatz

$$\delta\rho(t,x) = \widetilde{\delta\rho}(\omega,k)\,\mathrm{e}^{-\mathrm{i}(\omega t-kx)} \tag{VI.17}$$

in Eq. (VI.16b). This yields after some straightforward algebra the dispersion relation

$$\boxed{\omega^2 = c_s^2k^2 - \mathrm{i}\omega k^2\bar\nu.} \tag{VI.18}$$

Obviously, ω and k cannot be simultaneously (non-zero) real numbers.

[36] Introducing the kinetic shear resp. bulk viscosity coefficients ν resp. ν', one has $\bar\nu = \frac{4}{3}\nu + \nu'$, hence the notation.

Let us assume $k \in \mathbb{R}$ and $\omega = \omega_r + \mathrm{i}\omega_i$, where ω_r, ω_i are real. The dispersion relation becomes

$$\omega_r^2 - \omega_i^2 + 2\mathrm{i}\omega_r\omega_i = c_s^2 k^2 - \mathrm{i}\omega_r k^2 \bar{\nu} + \omega_i k^2 \bar{\nu},$$

which can only hold if both the real and imaginary parts are separately equal. The identity between the imaginary parts reads (for $\omega_r \neq 0$)

$$\omega_i = -\frac{1}{2}\bar{\nu}k^2, \tag{VI.19}$$

which is always negative, since $\bar{\nu}$ is non-negative. This term yields in the Fourier ansatz (VI.17) an exponentially decreasing factor $\mathrm{e}^{-\mathrm{i}(\mathrm{i}\omega_i)t} = \mathrm{e}^{-\bar{\nu}k^2 t/2}$ which represents the *damping* or *absorption* of the sound wave. The perturbations with larger wave number k, i.e. corresponding to smaller length scales, are more dampened that those with smaller k. This is quite natural, since a larger k also means a larger gradient, thus an increased influence of the viscous term in the Navier–Stokes equation.

In turn, the identity between the real parts of the dispersion relation yields

$$\omega_r^2 = c_s^2 k^2 - \frac{1}{4}\bar{\nu}^2 k^4. \tag{VI.20}$$

This gives for the phase velocity $c_\varphi \equiv \omega/k$ of the traveling waves

$$c_\varphi^2 = c_s^2 - \frac{1}{4}\bar{\nu}^2 k^2. \tag{VI.21}$$

That is, the "speed of sound" actually depends on its wave number k, and is smaller for small wavelength, i.e. high-k, perturbations—which are also those which are more damped out.

Relation (VI.21) also shows that the whole linear description adopted below Eqs. (VI.12) requires that the perturbations should have a relatively large wavelength, namely such that $k \ll 2c_s/\bar{\nu}$, so that c_φ remain real-valued. This is equivalent to requesting that the dissipative term $\bar{\nu}\triangle\delta\mathsf{v} \sim k^2\bar{\nu}\delta\mathsf{v}$ in the Navier–Stokes equation (VI.13b) should be much smaller than the term describing the local acceleration, $\partial_t\delta\mathsf{v} \sim \omega\delta\mathsf{v} \sim c_s k\delta\mathsf{v}$.

Remarks:

∗ Instead of considering "temporal damping" as was done above by assuming $k \in \mathbb{R}$ but $\omega \in \mathbb{C}$, one may investigate "spatial damping", i.e. assume $\omega \in \mathbb{R}$ and put the whole complex dependence in the wave number $k = k_r + \mathrm{i}k_i$. For (angular) frequencies ω much smaller than the inverse of the typical time scale $\tau_\nu \equiv \bar{\nu}/c_s^2$, one finds

$$\omega^2 \simeq c_s^2 k_r^2\left(1 + \frac{3}{4}\omega^2\tau_\nu^2\right) \quad \Leftrightarrow \quad c_\varphi \equiv \frac{\omega}{k_r} \simeq c_s\left(1 + \frac{3}{8}\omega^2\tau_\nu^2\right)$$

i.e. the phase velocity increases with the frequency, and on the other hand

$$k_i \simeq \frac{\bar{\nu}\omega^2}{2c_s^3}. \tag{VI.22}$$

The latter relation is known as *Stokes' law of sound attenuation*, k_i representing the inverse of the typical distance over which the sound wave amplitude decreases, due to the factor $\mathrm{e}^{\mathrm{i}(\mathrm{i}k_i)x} = \mathrm{e}^{-k_i x}$ in the Fourier ansatz (VI.17). Larger frequencies are thus absorbed on a smaller distance from the source of the sound wave.

Substituting $k = k_r + \mathrm{i}k_i = k_r(1 + \mathrm{i}\varkappa)$ in the dispersion relation (VI.18) and writing the identity of the real and imaginary parts, one obtains the system

$$\begin{cases} 2\varkappa = \omega\tau_\nu(1 - \varkappa^2) \\ \omega^2 = c_s^2 k_r^2(1 + 2\omega\tau_\nu\varkappa - \varkappa^2) \end{cases}$$

The first equation is a quadratic equation in $\varkappa$ that admits one positive and one negative solution: the latter can be rejected, while the former is $\varkappa \simeq \omega\tau_\nu/2 + \mathcal{O}\big((\omega\tau_\nu)^2\big)$. Inserting it in the second equation leads to the wanted results. □

An exact solution of the system of equations exists, yes it is neither enlightening mathematically, nor relevant from the physical point of view in the general case, as discussed in the next remark.

One may naturally also analyze the general case in which both ω and k are complex numbers. In any case, the phase velocity is given by $c_\varphi \equiv \omega/k_r$, although it is more difficult to recognize the physical content of the mathematical relations.

$*$ For air or water, the reduced kinetic viscosity $\bar{\nu}$ is of order 10^{-6}–10^{-5} $\mathrm{m}^2\cdot\mathrm{s}^{-1}$. With speeds of sound $c_s \simeq 300$–$1500\ \mathrm{m}\cdot\mathrm{s}^{-1}$, this yields typical time scales τ_ν of order 10^{-12}–10^{-10} s. That is, the change in the speed of sound (VI.21), or equivalently deviations from the assumption $\omega\tau_\nu \ll 1$ underlying the attenuation coefficient (VI.22), become relevant for sound waves in the gigahertz/terahertz regime(!). This explains why measuring the bulk viscosity is a non-trivial task.

The wavelengths $c_s\tau_\nu$ corresponding to the above frequencies τ_ν^{-1} are of order 10^{-9}–10^{-7} m. This is actually not far from the value of the mean free path in classical fluids, so that the whole description as a continuous medium starts being questionable.