Appendix C

Dimensional analysis

The purpose of this appendix is to introduce the so-called π-*theorem*, according to which a relation of the form

$$\mathcal{G} = f(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n) \tag{C.1}$$

between the mathematical representations $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$, ..., $\mathcal{G}_n$ of physical quantities can generically be simplified.

C.1 Buckingham π-theorem

C.1.1 Physical dimensions and dimensionally independent quantities

Before stating the theorem, we introduce a few definitions and notions.

As the reader most probably knows, physical quantities implicitly carry a (physical) *dimension*, which in particular dictates which units can be used to measure them. Given a quantity $\mathcal{G}$, the associated dimension is generally denoted $[\mathcal{G}]$.[35]

These physical dimensions can all be expressed as products of monomials of a handful of *base quantities*, namely length (symbol: L), mass (M), time (T), electric current (I), (thermodynamic) temperature (Θ), amount of substance (N), and luminous intensity (J). In fluid dynamics—and as long as one only considers uncharged fluids—, only the $q = 4$ quantities L, M, T and Θ are relevant, which is what we shall from now on assume.[36] For an arbitrary physical quantity $\mathcal{G}$, one may thus write

$$[\mathcal{G}] = \mathsf{L}^\alpha\, \mathsf{M}^\beta\, \mathsf{T}^\gamma\, \Theta^\delta \tag{C.2}$$

with rational coefficients α, β, γ, δ. In addition, quantities without physical dimension—such as pure numbers—are assigned the dimension $\mathsf{1}$.

A set of physical quantities $\mathcal{G}_1$, $\mathcal{G}_2$, ..., $\mathcal{G}_n$ are said to be *dimensionally independent* when the product $[\mathcal{G}_1]^{\lambda_1}[\mathcal{G}_2]^{\lambda_2}\cdots[\mathcal{G}_n]^{\lambda_n}$ is dimensionless if and only if every exponent λ_i vanishes:

$$[\mathcal{G}_1]^{\lambda_1}[\mathcal{G}_2]^{\lambda_2}\cdots[\mathcal{G}_n]^{\lambda_n} = \mathsf{1} \quad\Leftrightarrow\quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0. \tag{C.3}$$

[35] Note that we refer to $\mathcal{G}$ as the "physical quantity", while it should rather be called "mathematical representation of the physical quantity".

[36] The reader will be able to generalize the argumentation to $q \neq 4$ by herself/himself.

By definition, the base quantities are dimensionally independent of each other.

Introducing the dimensional exponents α_i, β_i, γ_i, δ_i for each quantity $\mathcal{G}_i$:

$$[\mathcal{G}_i] = \mathsf{L}^{\alpha_i}\,\mathsf{M}^{\beta_i}\,\mathsf{T}^{\gamma_i}\,\Theta^{\delta_i}, \tag{C.4}$$

one defines the *dimension matrix* associated with the set $\{\mathcal{G}_i\}_{i=1,\dots,n}$ as the $q \times n$ matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \delta_1 & \delta_2 & \cdots & \delta_n \end{pmatrix}. \tag{C.5}$$

One easily finds that n physical quantities $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ are dimensionally independent if and only if the rank r of the corresponding dimension matrix equals n—which obviously implies $n \leq q$.

Proof: The logarithm of the relation $[\mathcal{G}_1]^{\lambda_1}[\mathcal{G}_2]^{\lambda_2}\cdots[\mathcal{G}_n]^{\lambda_n} = 1$ reads, at least symbolically[37]

$$\lambda_1 \ln[\mathcal{G}_1] + \lambda_2 \ln[\mathcal{G}_2] + \cdots + \lambda_n \ln[\mathcal{G}_n] = 0.$$

Replacing each physical dimension $[\mathcal{G}_i]$ by its expression (C.4) in terms of the basis quantities, one obtains the linear system

$$\begin{cases} \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n = 0 \\ \beta_1\lambda_1 + \cdots + \beta_n\lambda_n = 0 \\ \gamma_1\lambda_1 + \cdots + \gamma_n\lambda_n = 0 \\ \delta_1\lambda_1 + \cdots + \delta_n\lambda_n = 0, \end{cases} \tag{C.6}$$

which can be recast in a matrix form involving the dimension matrix (C.5). Using basic results from linear algebra, the vector space spanned by the vectors $(\lambda_1, \dots, \lambda_n)^\mathsf{T}$ that satisfy the system is of dimension $n - r$, i.e. that space reduces to the zero vector iff $n = r$. □

If the rank r of the dimension matrix (C.5) is smaller than n, then the physical quantities $\mathcal{G}_1$, $\mathcal{G}_2$, …, $\mathcal{G}_n$ are not dimensionally independent, i.e. some of them can be expressed ("derived") in terms of the others.

Indeed, in that case, the linear system (C.6) is underdetermined. Given r independent coefficients among the $\{\lambda_i\}_{i=1,\dots,n}$, which up to relabeling can be chosen to be λ_1, …, λ_r, then the $n - r$ other coefficients λ_{r+1}, …, λ_n are linear combinations of the independent ones. Coming back to the physical quantities, $\{\mathcal{G}_1, \dots, \mathcal{G}_r\}$ form a "complete" set of dimensionally independent quantities, in terms of which the dimension of every quantity $\mathcal{G}_k$ with $k \in \{r+1, \dots, n\}$ can be expressed: there exist (rational) coefficients $a_{k,1}$, … $a_{k,r}$ such that

$$[\mathcal{G}_k] = [\mathcal{G}_1]^{a_{k,1}} \cdots [\mathcal{G}_r]^{a_{k,r}} \quad \text{for } k \in \{r+1, \dots, n\}. \tag{C.7}$$

Stated differently, the coefficients are such that the ratio

$$\pi_{k-r} \equiv \frac{\mathcal{G}_k}{\mathcal{G}_1^{a_{k,1}} \cdots \mathcal{G}_r^{a_{k,r}}} \quad \text{for } k \in \{r+1, \dots, n\} \tag{C.8}$$

is dimensionless: $[\pi_k] = 1$.

C.1.2 π-theorem

Let us come back to relation (C.1), with f some function. We assume that the n quantities $\mathcal{G}_1$, $\mathcal{G}_2$, …, $\mathcal{G}_n$ are *physically independent*, i.e. that the values they take can a priori be varied independently from each other. Denoting by r the rank of the dimension matrix associated with the set $\{\mathcal{G}_i\}_{i=1,\dots,n}$, we further assume that the first r quantities $\mathcal{G}_1$, …, $\mathcal{G}_r$ are *dimensionally* independent, while the dimensions of the remaining ones can be expressed by Eq. (C.7).

If relation (C.1) is physically meaningful, i.e. if it holds irrespective of the values (in a given system of units) of $\mathcal{G}$ and the quantities $\mathcal{G}_1$, …, $\mathcal{G}_n$, then necessarily $\mathcal{G}$ is not dimensionally independent

(37)… because taking the logarithm of a dimensionful quantity should upset you.

from the $\{\mathcal{G}_i\}_{i=1,\dots,n}$. One can thus find rational exponents $a_1, \dots, a_k$ such that

$$[\mathcal{G}] = [\mathcal{G}_1]^{a_1} \cdots [\mathcal{G}_r]^{a_r} \tag{C.9}$$

and accordingly define a dimensionless ratio

$$\pi \equiv \frac{\mathcal{G}}{\mathcal{G}_1^{a_1} \cdots \mathcal{G}_r^{a_r}}. \tag{C.10}$$

Relation (C.1) can then be rewritten as

$$\pi = \mathcal{G}_1^{a_1} \cdots \mathcal{G}_r^{a_r} f\big(\mathcal{G}_1, \dots, \mathcal{G}_r, \mathcal{G}_1^{a_{r+1,1}} \cdots \mathcal{G}_r^{a_{r+1,r}} \pi_1, \dots, \mathcal{G}_1^{a_{n,1}} \cdots \mathcal{G}_r^{a_{n,r}} \pi_{n-r}\big)$$

i.e., introducing an appropriate function $\mathcal{F}$, as

$$\pi = \mathcal{F}(\mathcal{G}_1, \dots, \mathcal{G}_r, \pi_1, \dots, \pi_{n-r}).$$

Now, since π is dimensionless, it is a number, whose value cannot depend on the system of units used to measure the dimensionful physical quantities $\mathcal{G}_1, \dots, \mathcal{G}_r$. Accordingly, the function $\mathcal{F}$ can actually not depend on its first r arguments, and one may replace it by a function f^* of the last $n-r$ arguments only and write

$$\pi = \mathsf{f}^*(\pi_1, \dots, \pi_{n-r}). \tag{C.11}$$

We have thus derived the *Buckingham π theorem*:

Any physically meaningful relation between $n+1$ dimensionful physical quantities $\mathcal{G}, \mathcal{G}_1, \dots, \mathcal{G}_n$ of the form $\mathcal{G} = f(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n)$ can be reduced to a relation $\pi = \mathsf{f}^*(\pi_1, \dots, \pi_{n-r})$ between $n+1-r$ dimensionless quantities, where r is the rank of the dimension matrix associated with the physical quantities $\mathcal{G}_1, \dots, \mathcal{G}_n$.

Remarks:

∗ The dimensionless quantities π, π_k are sometimes referred to as *Pi groups* or (less obscure, but more seldom) *similitude parameters*.

∗ The π theorem gives no information on the functional form of f^*.

C.2 Examples of application

Let us give a few examples of application of the π theorem in fluid dynamics.

C.2.1 Velocity of sea surface waves

We begin with examples corresponding to the physical situation discussed in Sec. ??. We consider a perfect fluid — characterized by its mass density ρ, dimension $\mathsf{M}\,\mathsf{L}^{-3}$ — in a constant gravitational field with acceleration g (dimension $\mathsf{L}\,\mathsf{T}^{-2}$). In the following, we want to find of the velocity c_w of waves — either phase or group velocity, it does not matter — depends on ρ, g, the wave number k ($[k] = \mathsf{L}^{-1}$) and other possibly relevant dimensionful parameters of the problem.

C.2.1 a Gravity-induced surface waves on an infinitely deep ocean

Since the fluid is assumed to be infinitely deep (and wide), there is no associated characteristic length scale. Accordingly, the velocity of the gravity-induced waves may be expected to depend on k, ρ, g only, i.e. there exists a function f of the $n = 3$ quantities such that

$$c_w = f(k, g, \rho)$$

which plays the role of relation (C.1). Note that the three quantities k, ρ, g are clearly dimensionally independent, so that the corresponding dimension matrix has rank $r = 3$.

Using k, ρ, g, the only combination with the dimension $\mathsf{L}\,\mathsf{T}^{-1}$ of a velocity is $\sqrt{g/k}$. The dimensionless "similitude parameter" associated with c_w [Eq. (C.10)] is thus $\pi = c_w/\sqrt{g/k}$.

Turning now to Eq. (C.11), we find that since here $n = r$ the relation formally reads

$$\frac{c_w}{\sqrt{g/k}} = \mathsf{f}^*() = \text{constant}, \tag{C.12a}$$

where $\mathsf{f}^*()$ designates a "function without argument". Reorganizing Eq. (C.12a) yields

$$c_w \propto \sqrt{\frac{g}{k}} \tag{C.12b}$$

which is indeed correct, see Eq. (**??**) (the purely numerical proportionality factor is 1 for the phase velocity, $\frac{1}{2}$ for the group velocity).

Remark: The mass density ρ plays no role here, nor in the following example. This could have been anticipated: in both cases there is no other physical quantity involved in the problem with a non-zero dimensional exponent for M, making it impossible to construct a dimensionless quantity involving ρ. This ultimately reflects the fact that the waves under consideration are induced by gravity only, so that the resulting acceleration of a fluid element is independent of its mass.

C.2.1 b Gravity-induced surface waves on an ocean with finite depth

Let us now consider the case of gravity induced waves on the surface of an ocean with depth h_0, which thus adds up to the list of physical quantities on which a wave velocity can depend, and we may write

$$c_w = f(k, g, \rho, h_0).$$

Since only 3 of the $n = 4$ quantities are dimensionally independent — say k, g and ρ again —, the similitude parameter $c_w/\sqrt{g/k}$ depends on $n - r = 1$ dimensionless parameter. The latter should involve the remaining dimensionful quantity h_0, and is quickly found to be kh_0, leading to

$$\frac{c_w}{\sqrt{g/k}} = \mathsf{f}^*(kh_0). \tag{C.13a}$$

This leads to

$$c_w \propto \sqrt{\frac{g}{k}}\, \mathsf{f}^*(kh_0). \tag{C.13b}$$

One can check starting from Eq. (**??**) that this is again correct — with $\mathsf{f}^* = \tanh$ for the phase velocity and a slightly more complicated form for the group velocity.

C.2.1 c Capillary waves on an infinitely deep ocean

Let us now discuss waves that are driven not by gravity, but by the surface tension σ (dimension $\mathsf{M}\,\mathsf{T}^{-2}$) at the water/air interface: σ represents the energy per unit surface[(38)] which is necessary to increase the size of the interface between the two fluids. When a flat ocean surface is deformed (by the action of wind), surface tension will tend to drive it back to flatness, to minimize the size of the interface, leading to so-called capillary waves.

In the case of an infinitely deep and wide ocean, the velocity of the capillary waves can depend on k, g, ρ — which will form our "basis" of dimensionally independent quantities —, and σ:

$$c_w = f(k, g, \rho, \sigma).$$

Using the basic quantities k, g, ρ, the unique combination with dimension $\mathsf{M}\,\mathsf{T}^{-2}$ is $\rho g/k^2$, so that relation (C.11) here reads

$$\frac{c_w}{\sqrt{g/k}} = \mathsf{f}^*\!\left(\frac{\sigma k^2}{\rho g}\right), \tag{C.14a}$$

[(38)] Energy has dimension $[E] = \mathsf{M}\,\mathsf{L}^2\,\mathsf{T}^{-2}$, surface L^2, leading at once to the dimension of σ.

or equivalently

$$c_w \propto \sqrt{\frac{g}{k}}\,\mathsf{f}^*\!\left(\frac{\sigma k^2}{\rho g}\right). \tag{C.14b}$$

If gravity is to play no role in the waves,[39] i.e. in the case of pure capillary waves, then g should drop out of the relation. This is only possible if f^* is (proportional to) the square root of its argument, leading to

$$c_w \propto \sqrt{\frac{\sigma k}{\rho}}.$$

Indeed, the dispersion relation for pure capillary waves under the conditions considered here is $\omega^2 = \sigma k^3/\rho$, leading to the above form for the phase and group velocities.

Remarks:

∗ The relative influence of gravity and surface tension on surface waves with wave vector k is quantified by a dimensionless number, the so-called Bond[al] or Eötvös number[am] Eo

$$\mathrm{Bo} = \mathrm{Eo} \equiv \frac{\rho g}{\sigma k^2}. \tag{C.15}$$

As could be expected, the argument of of the dimensionless function f^* is precisely (the inverse of) this number.

∗ Since σ is related to the interface between the water and the air, we should have considered not only the mass density ρ of the water, but also that of the air in the reasoning. Here we implicitly assumed that ρ_{air} is negligible, which reflects $\rho_{\mathrm{air}} \ll \rho_{\mathrm{water}}$. For the capillary waves at the boundary between two immiscible fluids — like water and oil — with similar mass densities, both of them should play a role in the velocity.

C.2.2 Expansion velocity of a shock front

To be completed...

[39]... apart from ensuring the flatness of the ocean in the absence of waves.

[al] W. N. Bond, 1897–1937 [am] L. Eötvös, 1848–1919