## V.2 Dynamic similarity

The incompressible motion of a Newtonian fluid is governed by the kinetic condition $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0$, the continuity equation (III.9), and the incompressible Navier–Stokes equation (III.33). In order to determine the relative influence of the various terms of the latter, it is often convenient to consider dimensionless forms of the equation, which leads to the introduction of a variety of dimensionless numbers.

For instance, the influence of the fluid mass density $\rho$ and shear viscosity $\eta$, which are uniform throughout the fluid, on a flow in the absence of volume forces is entirely encoded in the Reynolds number (§ V.2.1). Allowing for volume forces, either due to gravity or to inertial forces, their relative importance is controlled by similar dimensionless parameters (§ V.2.2).

Let $L_c$ resp. $\mathsf{v}_c$ be a characteristic length resp. velocity scale for a given flow. Since the Navier–Stokes equation itself does not involve any parameter with the dimension of a length or a velocity, both scales are controlled by "geometry", i.e. by the boundary conditions for the specific problem under consideration. Thus, $L_c$ may be the size (diameter, side length) of a tube in which the fluid flows or of an obstacle around which the fluid moves. In turn, $\mathsf{v}_c$ may be the uniform velocity far from such an obstacle.

With the help of $L_c$ and $\mathsf{v}_c$, one can rescale the physical quantities in the problem, so as to obtain dimensionless quantities, which will hereafter be denoted with $^*$:

$$\vec{r}^* \equiv \frac{\vec{r}}{L_c}, \quad \vec{\mathsf{v}}^* \equiv \frac{\vec{\mathsf{v}}}{\mathsf{v}_c}, \quad t^* \equiv \frac{t}{L_c/\mathsf{v}_c}, \quad \mathcal{P}^* \equiv \frac{\mathcal{P}-\mathcal{P}_0}{\rho\mathsf{v}_c^2}, \tag{V.10}$$

where $\mathcal{P}_0$ is some characteristic value of the (unscaled) pressure.

### V.2.1 Reynolds number

Consider first the incompressible Navier–Stokes equation in the absence of external volume forces. Rewriting it in terms of the dimensionless variables and fields (V.10) yields

$$\frac{\partial\vec{\mathsf{v}}^*(t^*,\vec{r}^*)}{\partial t^*} + \big[\vec{\mathsf{v}}^*(t^*,\vec{r}^*)\cdot\vec{\nabla}^*\big]\vec{\mathsf{v}}^*(t^*,\vec{r}^*) = -\vec{\nabla}^*\mathcal{P}^*(t^*,\vec{r}^*) + \frac{\eta}{\rho\mathsf{v}_c L_c}\triangle^*\vec{\mathsf{v}}^*(t^*,\vec{r}^*), \tag{V.11}$$

with $\vec{\nabla}^*$ resp. $\triangle^*$ the gradient resp. Laplacian with respect to the reduced position variable $\vec{r}^*$. Besides the reduced variables and fields, this equation involves a single dimensionless parameter, the inverse of the *Reynolds number*

$$\boxed{\mathrm{Re} \equiv \frac{\rho\mathsf{v}_c L_c}{\eta} = \frac{\mathsf{v}_c L_c}{\nu}.} \tag{V.12}$$

This number measures the relative importance of inertia and viscous friction forces on a fluid element or a body immersed in the moving fluid: at large resp. small Re, viscous effects are negligible resp. predominant.

**Remark:** As stated above Eq. (V.10), both $L_c$ and $\mathsf{v}_c$ are controlled by the geometry and boundary conditions. The Reynolds number—and every similar dimensionless we shall introduce hereafter—is thus a characteristic of a given flow, not of the fluid.

**Law of similitude**[li]

The solutions for the dynamical fields $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ at fixed boundary conditions and geometry—specified in terms of dimensionless ratios of geometrical lengths—are functions of the independent variables $t^*$, $\vec{r}^*$, and of the Reynolds number:

$$\vec{\mathsf{v}}^*(t^*,\vec{r}^*) = \vec{\mathsf{f}}_1^*(t^*,\vec{r}^*,\mathrm{Re}), \qquad \mathcal{P}^*(t^*,\vec{r}^*) = \mathsf{f}_2^*(t^*,\vec{r}^*,\mathrm{Re}), \tag{V.13}$$

with $\vec{\mathsf{f}}_1^*$ resp. $\mathsf{f}_2^*$ a vector resp. scalar function. The "physical" flow velocity and pressure fields are then given by

$$\vec{\mathsf{v}}(t,\vec{r}) = \mathsf{v}_c\vec{\mathsf{f}}_1^*\bigg(\frac{\mathsf{v}_c t}{L_c},\frac{\vec{r}}{L_c},\mathrm{Re}\bigg), \qquad \mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \rho\mathsf{v}_c^2\mathsf{f}_2^*\bigg(\frac{\mathsf{v}_c t}{L_c},\frac{\vec{r}}{L_c},\mathrm{Re}\bigg).$$

These equations underlie the use of fluid dynamical simulations with experimental models at a reduced scale, yet possessing the same (rescaled) geometry. Let $L_c$, $\mathsf{v}_c$ resp. $L_{\mathrm{M}}$, $\mathsf{v}_{\mathrm{M}}$ be the characteristic lengths and velocities of the real-size flow resp. of the reduced-scale experimental flow; for simplicity, we assume that the same fluid is used in both cases. If $\mathsf{v}_{\mathrm{M}}/\mathsf{v}_c = L_c/L_{\mathrm{M}}$, the Reynolds number for the experimental model is the same as for the real-size fluid motion: both flows then admit the same solutions $\vec{\mathsf{v}}^*$ and $\mathcal{P}^*$, and are said to be *dynamically similar*.

**Remark:** The functional relationships between the "dependent variables" $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ and the "independent variables" $t^*$, $\vec{r}^*$ and a dimensionless parameter (Re) represent a simple example of the more general (Vaschy[ae]–)Buckingham[af] $\pi$-theorem [22] in *dimensional analysis*, see Appendix C.

## V.2.2 Other dimensionless numbers

If the fluid motion is likely to be influenced by gravity, the corresponding volume force density $\vec{f}_V = -\rho\vec{g}$ (for a uniform gravity field) must be taken into account in the right member of the incompressible Navier–Stokes equation (III.33). Accordingly, if the latter is written in dimensionless form as in the previous paragraph, there is an additional term on the right hand side of Eq. (V.11), proportional to $1/\mathrm{Fr}^2$, with

$$\mathrm{Fr} \equiv \frac{\mathsf{v}_c}{\sqrt{gL_c}} \tag{V.14}$$

the *Froude number*.[ag] This dimensionless parameter measures the relative size of inertial and gravitational effects in the flow, the latter being important when Fr is small.

In the presence of gravity, the dimensionless dynamical fields $\vec{\mathsf{v}}^*$, $\mathcal{P}^*$ become functions of the reduced variables $t^*$, $\vec{r}^*$ controlled by both parameters Re and Fr.

The Navier–Stokes equation (III.32) holds in an inertial frame. In a non-inertial reference frame, there come additional terms, which may be expressed as fictive force densities on the right hand side, and come in addition to the "physical" volume force density $\vec{f}_V$. In the case of a reference frame in uniform rotation (with respect to an inertial frame) with angular velocity $\vec{\Omega}_0$, there are thus two extra contributions corresponding to centrifugal and Coriolis forces, namely $\vec{f}_{\mathrm{cent.}} = -\rho\vec{\nabla}\big[-\frac{1}{2}\big(\vec{\Omega}_0\times\vec{r}\big)^2\big]$ and $\vec{f}_{\mathrm{Cor.}} = -2\rho\vec{\Omega}_0\times\vec{\mathsf{v}}$, respectively.

[li] *Ähnlichkeitsgesetz*

[ae] A. Vaschy, 1857–1899 [af] E. Buckingham, 1867–1940 [ag] W. Froude, 1810–1879

The relative importance of the latter in a given flow can be estimated with dimensionless numbers. Thus, denoting $\Omega_0 \equiv |\vec{\Omega}_0|$, the *Ekman number*[(ah)]

$$\mathrm{Ek} \equiv \frac{\eta}{\rho\Omega_0 L_c^2} = \frac{\nu}{\Omega_0 L_c^2} \tag{V.15}$$

measures the relative size of (shear) viscous and Coriolis forces, with the latter predominating over the former when $\mathrm{Ek} \ll 1$.

One may also wish to compare the influences of the convective and Coriolis terms in the Navier–Stokes equation. This is done with the help of the *Rossby number*[(ai)]

$$\mathrm{Ro} \equiv \frac{\mathsf{v}_c}{\Omega_0 L_c} \tag{V.16}$$

which is small when the effect of the Coriolis force is the dominant one.

**Remark:** Quite obviously, the Reynolds (V.12), Ekman (V.15), and Rossby (V.16) numbers obey the simple identity

$$\mathrm{Ro} = \mathrm{Re} \cdot \mathrm{Ek}.$$

[(ah)] V. Ekman, 1874–1954 [(ai)] C.-G. Rossby, 1898–1957