IV.4 Potential flows

According to Lagrange's theorem (IV.20), every flow of a perfect barotropic fluid with conservative volume forces which is everywhere irrotational at a given instant remains irrotational at every time.

Focusing accordingly on the *incompressible* and irrotational motion of an ideal fluid with conservative volume forces, which is also referred to as a *potential flow*(xlvii), the dynamical equations can be recast such that the main one is a linear partial differential equation for the *velocity potential* (§ IV.4.1), for which there exist mathematical results (§ IV.4.2). Two-dimensional potential flows are especially interesting, since one may then introduce a complex velocity potential—and the corresponding complex velocity—, which is a holomorphic function (§ IV.4.3). This allows one to use the full power of complex analysis so as to devise flows around obstacles with various geometries by combining "elementary" solutions and deforming them.

IV.4.1 Equations of motion in potential flows

Using a known result from vector analysis, a vector field whose curl vanishes everywhere on a simply connected domain of $\mathbb{R}^3$ can be written as the gradient of a scalar field. Thus, in the case of an irrotational flow $\vec{\nabla} \times \vec{v}(t,\vec{r}) = \vec{0}$, the velocity field can be expressed as

$$\vec{v}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r}) \tag{IV.29}$$

with $\varphi(t,\vec{r})$ the so-called *velocity potential*.(xlviii)

Remarks:

∗ The minus sign in definition (IV.29) is purely conventional. While the choice adopted here is not universal, it has the advantage of being directly analogous to the convention in electrostatics ($\vec{E} = -\vec{\nabla}\Phi_{\text{Coul.}}$) or Newtonian gravitation physics ($\vec{g} = -\vec{\nabla}\Phi_{\text{Newt.}}$).

∗ Since Lagrange's theorem does not hold in a dissipative fluid, in which vorticity can be created or annihilated (Sec. ??), the rationale behind the definition of the velocity potential disappears.

Using the velocity potential (IV.29) and the relation $\vec{a}_V = -\vec{\nabla}\Phi$ expressing that the volume forces are conservative, the Euler equation (III.20) reads

$$-\frac{\partial\vec{\nabla}\varphi(t,\vec{r})}{\partial t} + \vec{\nabla}\left\{\frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2} + \Phi(t,\vec{r})\right\} = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}).$$

Assuming that the flow is also incompressible, and thus ρ constant, this becomes

$$-\frac{\partial\vec{\nabla}\varphi(t,\vec{r})}{\partial t} + \vec{\nabla}\left\{\frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2} + \frac{\mathcal{P}(t,\vec{r})}{\rho} + \Phi(t,\vec{r})\right\} = \vec{0}. \tag{IV.30}$$

or equivalently

$$-\frac{\partial\varphi(t,\vec{r})}{\partial t} + \frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2} + \frac{\mathcal{P}(t,\vec{r})}{\rho} + \Phi(t,\vec{r}) = C(t), \tag{IV.31}$$

where $C(t)$ denotes a function of time only.

Eventually, expressing the incompressibility condition [cf. Eq. (II.15)] $\vec{\nabla}\cdot\vec{v}(t,\vec{r}) = 0$ leads to the *Laplace equation*(w) for the velocity potential φ

$$\triangle\varphi(t,\vec{r}) = 0. \tag{IV.32}$$

(xlvii) *Potentialströmung* (xlviii) *Geschwindigkeitspotential*

(w) P.-S. (de) Laplace, 1749–1827

Equations (IV.29), (IV.31) and (IV.32) are the three equations of motion governing potential flows. Since the Laplace equation is partial differential, it is still necessary to specify the corresponding boundary conditions.

In agreement with the discussion in § III.3.2 c, there are two types of conditions: at walls or obstacles, which are impermeable to the fluid; and "at infinity"—for a flow in an unbounded domain of space—, where the fluid flow is generally assumed to be uniform. Choosing a proper reference frame $\mathcal{R}$, this uniform motion of the fluid may be turned into having a fluid at rest. Denoting by $\mathcal{S}(t)$ the material surface associated with walls or obstacles, which are assumed to be moving with velocity $\vec{v}_{\text{obs.}}$ with respect to $\mathcal{R}$, and by $\vec{e}_n(t,\vec{r})$ the unit normal vector to $\mathcal{S}(t)$ at a given point $\vec{r}$, the condition of vanishing relative normal velocity reads

$$-\vec{e}_n(t,\vec{r})\cdot\vec{\nabla}\varphi(t,\vec{r}) = \vec{e}_n(t,\vec{r})\cdot\vec{v}_{\text{obs.}}(t,\vec{r}) \quad \text{on} \quad \mathcal{S}(t). \tag{IV.33a}$$

In turn, the condition of rest at infinity reads

$$\varphi(t,\vec{r}) \underset{|\vec{r}|\to\infty}{\sim} K(t), \tag{IV.33b}$$

where in practice the scalar function $K(t)$ will be given.

Remarks:

∗ Since the Laplace equation (IV.32) is linear—the non-linearity of the Euler equation is in Eq. (IV.31), which becomes trivial once the spatial dependence of the velocity potential has been determined—, it will be possible to *superpose* the solutions of "simple" problems to obtain the solution for a more complicated geometry.

∗ In potential flows, the dependences on time and space are somewhat separated: The Laplace equation (IV.32) governs the spatial dependence of φ and thus $\vec{\mathsf{v}}$; meanwhile, time enters the boundary conditions (IV.33), and is thus used to fix the amplitude of the solution of the Laplace equation. In turn, when φ is known, relation (IV.31) gives the pressure field, where the integration "constant" $C(t)$ will also be fixed by boundary conditions.

IV.4.2 Mathematical results on potential flows

The *boundary value problem* consisting of the Laplace differential equation (IV.32) together with the boundary conditions on normal derivatives (IV.33) is called a *Neumann problem*[(x)] or boundary value problem of the second kind. For such problems, results on the existence and unicity of solutions have been established, which we shall now state without further proof.[(21)]

IV.4.2 a Potential flows in simply connected regions

The simplest case is that of a potential flow on a simply connected domain $\mathscr{D}$ of space. $\mathscr{D}$ may be unbounded, provided the condition at infinity is that the fluid should be at rest, Eq. (IV.33b).

> *On a simply connected domain, the Neumann problem (IV.32)–(IV.33) for the velocity potential admits a solution $\varphi(t,\vec{r})$, which is unique up to an additive constant. In turn, the flow velocity field $\vec{\mathsf{v}}(t,\vec{r})$ given by relation (IV.29) is unique.* (IV.34)

For a flow on a simply connected region, the relation (IV.29) between the flow velocity and its potential is "easily" invertible: fixing some reference position $\vec{r}_0$ in the domain, one may write

$$\varphi(t,\vec{r}) = \varphi(t,\vec{r}_0) - \int_{\vec{\gamma}} \vec{\mathsf{v}}(t,\vec{r}\,')\cdot \mathrm{d}\vec{\ell}(\vec{r}\,') \tag{IV.35}$$

where the line integral is taken along *any* path $\vec{\gamma}$ on $\mathscr{D}$ connecting the positions $\vec{r}_0$ and $\vec{r}$.

(21) The Laplace differential equation is dealt with in many textbooks, as e.g. in Ref. [18, Chapters 7–9], [19, Chapter 4], or [20, Chapter VII].

(x) C. Neumann, 1832–1925

> That the line integral only depends on the path extremities $\vec{r}_0$, $\vec{r}$, not on the path itself, is clearly equivalent to Stokes' theorem stating that the circulation of velocity along *any* closed contour in the domain $\mathscr{D}$ is zero—it equals the flux of the vorticity, which is everywhere zero, through a surface delimited by the contour and entirely contained in $\mathscr{D}$.

Thus, $\varphi(t,\vec{r})$ is uniquely defined once the value $\varphi(t,\vec{r}_0)$, which is the arbitrary additive constant mentioned above, has been fixed.

This reasoning no longer holds in a multiply connected domain, as we now further discuss.

IV.4.2 b Potential flows in doubly connected regions

As a matter of fact, in a doubly (or a fortiori multiply) connected domain, there are by definition non-contractible closed paths. Consider for instance the domain $\mathscr{D}$ traversed by an infinite cylinder—which is not part of the domain—of Fig. IV.7. The path going from $\vec{r}_0$ to $\vec{r}_2$ along $\vec{\gamma}_{0\to 2}$ and coming back to $\vec{r}_0$ along $\vec{\gamma}\,'_{0\to 2}$[22] cannot be continuously shrunk to a point without leaving $\mathscr{D}$. This opens the possibility that the line integral in relation (IV.35) could depend on the path connecting two points.

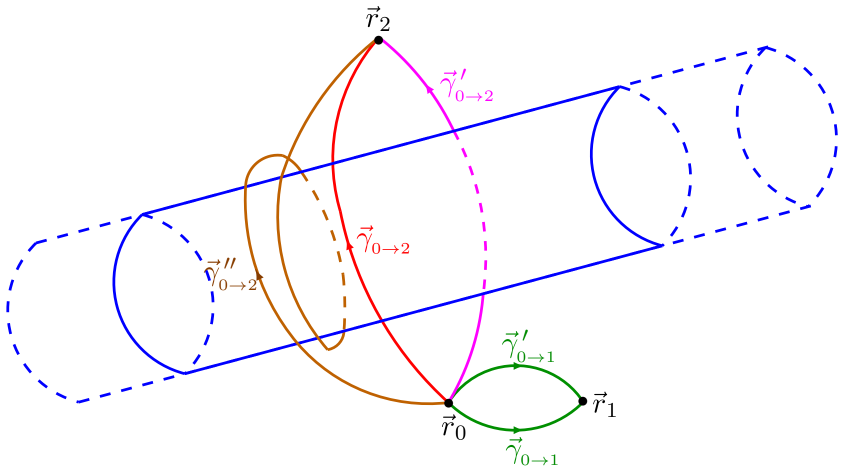

Figure IV.7

In a doubly connected domain $\mathscr{D}$, there is only a single "hole" that prevents closed paths from being homotopic to a point, i.e. contractible. Let $\Gamma(t)$ denote the circulation at time t of the velocity around a closed contour, with a given "positive" orientation, circling the hole *once*. One easily checks—e.g. invoking Stokes' theorem—that this circulation has the same value for all closed paths going only once around the hole with the same orientation, since they can be continuously deformed into each other without leaving $\mathscr{D}$. Accordingly, the "universal" value of the circulation $\Gamma(t)$ is also referred to as *cyclic constant* of the flow.

More generally, the circulation at time t of the velocity around a closed curve circling the hole n times and oriented in the positive resp. negative direction is $n\Gamma(t)$ resp. $-n\Gamma(t)$.

Going back to the line integral in Eq. IV.35, its value will generally depend on the path $\vec{\gamma}$ from $\vec{r}_0$ to $\vec{r}$—or more precisely, on the class, defined by the number of loops around the hole, of the path. Illustrating this idea on Fig. IV.7, while the line integral from $\vec{r}_0$ to $\vec{r}_2$ along the path $\vec{\gamma}_{0\to 2}$ will have a given value $\mathcal{I}$, the line integral along $\vec{\gamma}\,'_{0\to 2}$ will differ by one (say, positive) unit of $\Gamma(t)$ and be equal to $\mathcal{I}+\Gamma(t)$. In turn, the integral along $\vec{\gamma}\,''_{0\to 2}$, which makes one negatively oriented loop more than $\vec{\gamma}_{0\to 2}$ around the cylinder, takes the value $\mathcal{I}-\Gamma(t)$.

These preliminary discussions suggest that if the Neumann problem (IV.32)–(IV.33) for the velocity potential on a doubly connected domain admits a solution $\varphi(t,\vec{r})$, the latter will not be

[22] More precisely, if $\vec{\gamma}\,'_{0\to 2}$ is parameterized by $\lambda\in[0,1]$ when going from $\vec{r}_0$ to $\vec{r}_2$, a path from $\vec{r}_2$ to $\vec{r}_0$ with the same geometric support—which is what is meant by "coming back along $\vec{\gamma}\,'_{0\to 2}$"—is $\lambda\mapsto\vec{\gamma}\,'_{0\to 2}(1-\lambda)$.

a scalar function in the usual sense, but rather a *multivalued* function, whose various values at a given position $\vec{r}$ at a fixed time t differ by an integer factor of the cyclic constant $\Gamma(t)$.

All in all, the following result holds *provided the cyclic constant* $\Gamma(t)$ *is known*, i.e. if its value at time t is part of the boundary conditions:

> *On a doubly connected domain, the Neumann problem (IV.32)–(IV.33) for the velocity potential with given cyclic constant* $\Gamma(t)$ *admits a solution* $\varphi(t,\vec{r})$*, which is unique up to an additive constant. The associated flow velocity field* $\vec{\mathsf{v}}(t,\vec{r})$ *is unique.* (IV.36)

The above wording does not specify the nature of the solution $\varphi(t,\vec{r})$:

- if $\Gamma(t) = 0$, in which case the flow is said to be *acyclic*, the velocity potential $\varphi(t,\vec{r})$ is a univalued function;
- if $\Gamma(t) \neq 0$, i.e. in a *cyclic flow*, the velocity potential $\varphi(t,\vec{r})$ is a multivalued function of its spatial argument. Yet as the difference between the various values at a given $\vec{r}$ is function of time only, the velocity field (IV.29) remains uniquely defined.

Remarks:

∗ Inspecting Eq. (IV.31), one might fear that the pressure field $\mathcal{P}(t,\vec{r})$ could be multivalued, reflecting the term $\partial\varphi(t,\vec{r})/\partial t$. Actually, however, Eq. (IV.31) is a first integral of Eq. (IV.30), in which the $\vec{r}$-independent multiples of $\Gamma(t)$ distinguishing the multiple values of $\varphi(t,\vec{r})$ disappear when the gradient is taken. That is, the term $\partial\varphi(t,\vec{r})/\partial t$ is to be taken with a grain of salt, since in fact it does not contain $\Gamma(t)$ nor its time derivative.

∗ In agreement with the first remark, the reader should remember that the velocity potential $\varphi(t,\vec{r})$ is just a useful auxiliary mathematical function,[(23)] yet the physical quantity is the velocity itself. Thus the possible multivaluedness of $\varphi(t,\vec{r})$ is not a real physical problem.

[(23)] Like its cousins: gravitational potential $\Phi_{\text{Newt.}}$, electrostatic potential $\Phi_{\text{Coul.}}$, magnetic vector potential $\vec{A}$…