IV.1.2 Incompressible fluid

Consider first an incompressible fluid—or, more correctly, a fluid whose compressibility can as a first approximation be neglected—with constant, uniform mass density ρ.
The fundamental equation of hydrostatics (IV.2) in the uniform gravitational field $-g\,\vec{e}_z$ then yields

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial x} = \frac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0, \qquad \frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\rho g.$$

That is, one recovers Pascal's(ac) law

$$\mathcal{P}(\vec{r}) = \mathcal{P}(z) = \mathcal{P}_0 - \rho g z, \tag{IV.4}$$

with $\mathcal{P}_0$ the pressure at the reference point with altitude $z = 0$.

For instance, the reader is possibly aware that at a depth of 10 meters under (liquid) water ($\rho = 10^3$ kg·m^{-3}), the pressure is

$$\mathcal{P}(-10\,\mathrm{m}) = \mathcal{P}(0) + 10^3 \cdot g \cdot 10 \approx 2\times 10^5\ \mathrm{Pa},$$

with $\mathcal{P}(0) \approx 10^5$ Pa the typical atmospheric pressure at sea level.

IV.1.3 Fluid at global thermal equilibrium

To depart from the assumption of incompressibility, whose range of validity is quite limited, let us instead consider a fluid at global thermal equilibrium i.e. with a uniform temperature T; for instance, an ideal gas, obeying at each point the mechanical equation of state $\mathcal{P}(\vec{r}) = n(\vec{r})k_{\mathrm{B}}T$, where n denotes the number density.

Denoting by m the mass of a molecule of that gas, the mass density is related to pressure and temperature by $\rho = m\mathcal{P}/k_{\mathrm{B}}T$, so that Eq. (IV.2) reads

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial x} = \frac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0, \qquad \frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\frac{mg}{k_{\mathrm{B}}T}\,\mathcal{P}(\vec{r}),$$

i.e. one obtains the *barometric formula*(xlvii)

$$\mathcal{P}(\vec{r}) = \mathcal{P}(z) = \mathcal{P}_0 \exp\left(-\frac{mgz}{k_{\mathrm{B}}T}\right).$$

(xlvii) *barometrische Höhenformel*

(ac) B. Pascal, 1623–1662

Invoking the equation of state, one sees that the molecule number density $\mathcal{n}(\vec{r})$ is also exponentially distributed, in agreement with the Maxwell[(ad)] distribution of statistical mechanics since mgz is the potential gravitational energy of a molecule at altitude z.

Taking as example air—which is a fictive ideal gas with molar mass[(12)] $\mathcal{N}_A\, m_{\rm air} = 29\ \text{g}\cdot\text{mol}^{-1}$—the ratio $k_{\rm B}T/m_{\rm air}g$ equals 8.8×10^3 m for $T = 300$ K, i.e. the pressure drops by a factor 2 for every elevation gain of ca. 6 km. Obviously, however, assuming a constant temperature in the Earth atmosphere over such a length scale is unrealistic.

IV.1.4 Isentropic fluid

Let us now assume that the entropy per particle is constant throughout the perfect fluid at rest under study: $s/\mathcal{n} =$ constant, with s the entropy density and $\mathcal{n}$ the particle number density.

> We shall show in § ?? that the ratio $s/\mathcal{n}$ is always conserved in the motion of a relativistic perfect fluid. Taking the low-velocity limit, one deduces the conservation of $s/\mathcal{n}$ in a non-relativistic non-dissipative flow: $\mathrm{D}(s/\mathcal{n})/\mathrm{D}t = 0$, implying that $s/\mathcal{n}$ is constant along pathlines, i.e. in the stationary regime along streamlines. Here we assume that $s/\mathcal{n}$ takes the same value everywhere.

Consider now the enthalpy $H = U + \mathcal{P}\mathcal{V}$ of the fluid, whose change in an infinitesimal process is the differential $\mathrm{d}H = T\,\mathrm{d}S + \mathcal{V}\,\mathrm{d}\mathcal{P} + \mu\,\mathrm{d}N$.[(13)] In this relation, μ denotes the chemical potential, which will however play no further role as we assume that the number of molecules in the fluid is constant, leading to $\mathrm{d}N = 0$. Dividing by N gives

$$\mathrm{d}\left(\frac{H}{N}\right) = T\,\mathrm{d}\left(\frac{S}{N}\right) + \frac{\mathcal{V}}{N}\,\mathrm{d}\mathcal{P},$$

or equivalently, in terms of the respective densities

$$\mathrm{d}\left(\frac{w}{\mathcal{n}}\right) = T\,\mathrm{d}\left(\frac{s}{\mathcal{n}}\right) + \frac{1}{\mathcal{n}}\,\mathrm{d}\mathcal{P},$$

where w denotes the (volumetric) enthalpy density. Dropping the first term on the right-hand side, since $s/\mathcal{n}$ is assumed to be constant, and dividing by the mass of a molecule of the fluid, one finds

$$\mathrm{d}\left(\frac{w}{\rho}\right) = \frac{1}{\rho}\,\mathrm{d}\mathcal{P}. \tag{IV.5}$$

This identity relates the change in specific enthalpy w/ρ to the pressure variation $\mathcal{P}$ in an elementary process at constant particle number N and constant entropy per particle $s/\mathcal{n}$. If one considers a fluid at local thermodynamic equilibrium, in which w/ρ and $\mathcal{P}$ take different values at different places, the identity relates the difference in w/ρ to that in $\mathcal{P}$ between two (neighboring) points. Dividing by the distance between the two points, and in the limit where this distance vanishes, one derives an identity similar to (IV.5) with gradients instead of differentials:

$$\vec{\nabla}\left[\frac{w(\vec{r})}{\rho(\vec{r})}\right] = \frac{1}{\rho(\vec{r})}\,\vec{\nabla}\mathcal{P}(\vec{r})$$

Together with the fundamental equation of hydrostatics Eq. (IV.2), one thus obtains

$$\vec{\nabla}\left[\frac{w(\vec{r})}{\rho(\vec{r})} + \Phi(\vec{r})\right] = \vec{0} \tag{IV.6}$$

that is $\dfrac{w(z)}{\rho(z)} + gz =$ constant in the case of a homogeneous gravity field $-g\,\vec{e}_z$.

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(12) $\mathcal{N}_A$ denotes the Avogadro[(ae)] number.

(13) The reader in need of a short reminder on thermodynamics is referred to Appendix ??.

(ad) J. C. Maxwell, 1831–1879 (ae) A. Avogadro, 1776–1856

Taking as example an ideal diatomic gas, its internal energy is $U = \frac{5}{2}Nk_{\rm B}T$, resulting in the enthalpy density

$$w = e + \mathcal{P} = \frac{5}{2}nk_{\rm B}T + nk_{\rm B}T = \frac{7}{2}nk_{\rm B}T.$$

That is, $\dfrac{w}{\rho} = \dfrac{7}{2}\dfrac{k_{\rm B}T}{m}$, with m the mass of a molecule of gas. Equation (IV.6) then gives

$$\frac{{\rm d}T(z)}{{\rm d}z} = -\frac{mg}{\frac{7}{2}k_{\rm B}}.$$

In the case of air, the term on the right hand side equals $9.77 \times 10^{-3}\ {\rm K\cdot m^{-1}} = 9.77\ {\rm K\cdot km^{-1}}$, i.e. the temperature drops by ca. 10 degrees for an elevation gain of 1 km. This represents a much better modeling of the (lower) Earth atmosphere than the isothermal assumption of § IV.1.3.

Remarks:

∗ The *International Standard Atmosphere* (ISA)(14) model of the Earth atmosphere assumes a (piecewise) linear dependence of the temperature on the altitude. The adopted value of the temperature gradient in the troposphere is smaller than the above, namely $6.5\ {\rm K\cdot km^{-1}}$, to take into account the possible condensation of water vapor into droplets or even ice.

∗ Coming back to the derivation of relation (IV.6), if we had not assumed s/n constant, we would have found the relation

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = \vec{\nabla}\left[\frac{w(\vec{r})}{\rho(\vec{r})}\right] - T(\vec{r})\,\vec{\nabla}\left[\frac{s(\vec{r})}{\rho(\vec{r})}\right], \qquad \text{(IV.7)}$$

which we shall use in § IV.2.1.

IV.2 Steady inviscid flows

We now turn to stationary solutions of the equations of motion for perfect fluids: all partial time derivatives vanish—and accordingly we shall drop the t variable—, yet the flow velocity $\vec{\rm v}(\vec{r})$ may now be non-zero. Under those conditions, the equations (III.21) and (III.38) expressing the conservations of momentum and energy collapse onto a single equation (§ IV.2.1). Some applications of the latter in the particular case of an incompressible flow are then presented (§ IV.2.2).

IV.2.1 Bernoulli equation

Replacing in the Euler equation (III.23) the pressure term with the help of relation (IV.7) and the acceleration due to volume forces by its expression in terms of the potential energy per unit mass, one finds

$$\frac{\partial\vec{\rm v}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{\rm v}(t,\vec{r})^2}{2}\right] - \vec{\rm v}(t,\vec{r})\times\vec{\omega}(t,\vec{r}) = T(t,\vec{r})\,\vec{\nabla}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right] - \vec{\nabla}\left[\frac{w(t,\vec{r})}{\rho(t,\vec{r})}\right] - \vec{\nabla}\Phi(t,\vec{r}), \qquad \text{(IV.8)}$$

which is rather more clumsy than the starting point (III.23), due to the many thermodynamic quantities it involves on its right hand side. Gathering all gradient terms together, one obtains

$$\frac{\partial\vec{\rm v}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{\rm v}(t,\vec{r})^2}{2} + \frac{w(t,\vec{r})}{\rho(t,\vec{r})} + \Phi(t,\vec{r})\right] = \vec{\rm v}(t,\vec{r})\times\vec{\omega}(t,\vec{r}) + T(t,\vec{r})\,\vec{\nabla}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right]. \qquad \text{(IV.9)}$$

In the stationary regime, the first term on the left-hand side disappears(15)—and we now omit the time variable from the equations.

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(14) See e.g. `https://en.wikipedia.org/wiki/International_Standard_Atmosphere`.

(15) This yields a relation known as *Crocco's*(af) *theorem*(xlviii)

(xlviii) *Croccos Wirbelsatz*

(af) L. Crocco, 1909–1986

Let $\mathrm{d}\vec{\ell}(\vec{r})$ denote a vector tangential to the streamline at position $\vec{r}$, i.e. parallel to $\vec{v}(\vec{r})$. In the scalar product of $\mathrm{d}\vec{\ell}(\vec{r})$ with Eq. (IV.9), both terms on the right hand side yield zero:

- the mixed product $\mathrm{d}\vec{\ell}(\vec{r}) \cdot [\vec{v}(\vec{r}) \times \vec{\omega}(\vec{r})]$ is zero for it involves two collinear vectors;
- $\mathrm{d}\vec{\ell}(\vec{r}) \cdot \vec{\nabla}[s(\vec{r})/\rho(\vec{r})]$ vanishes due to the conservation of s/n in flows of perfect fluids, which together with the stationarity reads $\vec{v}(\vec{r}) \cdot \vec{\nabla}[s(\vec{r})/n(\vec{r})] = 0$, where n is proportional to ρ.

In turn, $\mathrm{d}\vec{\ell}(\vec{r}) \cdot \vec{\nabla}$ represents the derivative along the direction of $\mathrm{d}\vec{\ell}(\vec{r})$, i.e. along the streamline at $\vec{r}$. Thus, the derivative of the term in squared brackets on the left hand side of Eq. (IV.9) vanishes along a streamline, i.e. the term remains constant on a streamline:

$$\boxed{\frac{\vec{v}(\vec{r})^2}{2} + \frac{w(\vec{r})}{\rho(\vec{r})} + \Phi(\vec{r}) = \text{constant along a streamline}} \tag{IV.10}$$

where the value of the constant depends on the streamline. Relation (IV.10) is referred to as the *Bernoulli*[ag] *equation.*

In the stationary regime, the energy conservation equation (III.38), in which one recognizes the enthalpy density $w(\vec{r}) = e(\vec{r}) + \mathcal{P}(\vec{r})$ in the flux term, leads to the same relation (IV.10).

The first term in Eq. (III.38) vanishes due to the stationarity condition, leaving (we drop the variables)

$$\vec{\nabla} \cdot \left[\left(\frac{\vec{v}^2}{2} + \frac{w}{\rho} + \Phi\right)\rho\,\vec{v}\right] = 0.$$

Applying the product rule to the left member, one finds a first term proportional to $\vec{\nabla} \cdot (\rho\,\vec{v})$—which vanishes thanks to the continuity equation (III.12)—, leaving only the other term, which is precisely ρ times the derivative along $\vec{v}$ of the left hand side of the Bernoulli equation. □

Bernoulli equation in particular cases

Coming back to Eq. (IV.9), if the steady flow is irrotational, i.e. $\vec{\omega}(\vec{r}) = \vec{0}$ everywhere, and isentropic, i.e. $s(\vec{r})/n(\vec{r})$ is uniform, then the gradient on the left hand side vanishes. That is, the constant in the Bernoulli equation (IV.10) is independent of the streamline, i.e. it is the same everywhere.

In case the flow is incompressible, i.e. $\vec{\nabla} \cdot \vec{v}(\vec{r}) = 0$, then the continuity equation shows that the mass density ρ becomes uniform throughout the fluid. One may then pull the factor $1/\rho$ inside the pressure gradient in the Euler equation (III.23). Repeating then the same steps as below Eq. (IV.9), one finds that the Bernoulli equation now reads

$$\boxed{\text{In incompressible flows } \ \frac{\vec{v}(\vec{r})^2}{2} + \frac{\mathcal{P}(\vec{r})}{\rho} + \Phi(\vec{r}) \ \text{ is constant along a streamline.}} \tag{IV.11}$$

This is the form which we shall use in the applications hereafter.

Can this form be reconciled with the other one (IV.10), which is still what follows from the energy conservation equation? Subtracting one from the other, one finds that the ratio $e(\vec{r})/\rho$ is constant along streamlines. That is, since ρ is uniform, the internal energy density is constant along pathlines—which coincide with streamlines in a steady flow. Now, thermodynamics expresses the differential $\mathrm{d}e$ through $\mathrm{d}s$ and $\mathrm{d}n$: since both entropy and particle number are conserved along a pathline, so is internal energy, i.e. Eq. (IV.10) is compatible with Eq. (IV.11).

[ag] J. Bernoulli, 1655–1705