III.3.3 Newtonian fluid: Navier–Stokes equation

In a real moving fluid, there are *friction forces* that contribute to the transport of momentum between neighboring fluid layers when the latter are in relative motion. Accordingly, the momentum flux-density tensor is no longer given by Eq. (III.21b) or (III.22), but now contains extra terms, involving derivatives of the flow velocity. The Euler equation must therefore be replaced by an alternative dynamical equation, including the friction forces.

III.3.3 a Momentum flux density in a Newtonian fluid

The momentum flux density (III.21b) in a perfect fluid only contains two terms—one proportional to $\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})$, the other proportional to the components g^{ij} of the inverse metric tensor. Since the coefficients in front of these two terms could a priori depend on $\vec{\mathsf{v}}^2$, this represents the most general symmetric tensor of degree 2 which can be constructed with the help of the flow velocity only.

If the use of terms that depend on the spatial derivatives of the velocity field is also allowed, the components of the momentum flux-density tensor can be of the following form, where for the sake of brevity the variables t and $\vec{r}$ are omitted

$$\mathsf{T}^{ij} = \rho \mathsf{v}^i \mathsf{v}^j + \mathcal{P} g^{ij} + A\frac{\mathrm{d}\mathsf{v}^i}{\mathrm{d}x_j} + B\frac{\mathrm{d}\mathsf{v}^j}{\mathrm{d}x_i} + \mathcal{O}\!\left(\frac{\mathrm{d}^2\mathsf{v}_i}{\mathrm{d}x_j\,\mathrm{d}x_k}\right) + \cdots, \tag{III.26}$$

with coefficients A, B that depend on i, j and on the fluid under consideration.

> This ansatz for T^{ij}, as well as the form of the energy flux density involved in Eq. (III.37) below, can be "justified" by starting from a microscopic *kinetic theory* of the fluid and writing the solutions of the corresponding equation of motion as a specific expansion—which turns out to be in powers of the Knudsen number (I.4). This also explains why terms of the type $\mathsf{v}^i \partial \mathcal{P}/\partial x_j$ or $\mathsf{v}^i \partial T/\partial x_j$, with T the temperature, were not considered in Eq. (III.26).
>
> Despite these theoretical considerations, in the end the actual justification for the choices of momentum or energy flux density is the agreement with the measured behavior of fluids.

As discussed in § I.1.3, the description of a system of particles as a continuous medium, and in particular as a fluid, in local thermodynamic equilibrium, rests on the assumption that the macroscopic quantities of relevance for the medium vary slowly both in space and time. Accordingly, (spatial) gradients should be small: the third and fourth terms in Eq. (III.26) should thus be on the one hand much smaller than the first two ones, on the other hand much larger than the rightmost term as well as those involving higher-order derivatives or of powers of the first derivatives. Neglecting these smaller terms, one obtains "first-order dissipative fluid dynamics", which describes the motion of *Newtonian fluids*—this actually defines the latter.

Using the necessary symmetry of T^{ij}, the third and fourth terms in Eq. (III.26) can be rewritten as the sum of a traceless symmetric contribution and a tensor proportional to the inverse metric tensor. This leads to the momentum flux-density tensor

$$\mathsf{T}^{ij}(t,\vec{r}) = \rho(t,\vec{r})\mathsf{v}^i(t,\vec{r})\mathsf{v}^j(t,\vec{r}) + \mathcal{P}(t,\vec{r})\,g^{ij}(t,\vec{r})$$
$$- \eta(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}^i(t,\vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}\mathsf{v}^j(t,\vec{r})}{\mathrm{d}x_i} - \frac{2}{3}g^{ij}(t,\vec{r})\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right] \tag{III.27a}$$
$$- \zeta(t,\vec{r})g^{ij}(t,\vec{r})\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}).$$

In geometric formulation, this reads

$$\boxed{\mathsf{T}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})\otimes\vec{\mathsf{v}}(t,\vec{r}) + \mathcal{P}(t,\vec{r})\,\mathsf{g}^{-1}(t,\vec{r}) + \boldsymbol{\pi}(t,\vec{r})} \tag{III.27b}$$

where dissipative effects are encoded in the *viscous stress tensor*[(xxxvii)]

$$\boxed{\text{for a Newtonian fluid:}\\ \boldsymbol{\pi}(t,\vec{r}) \equiv -2\eta(t,\vec{r})\left[\mathsf{D}(t,\vec{r}) - \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\mathsf{g}^{-1}(t,\vec{r})\right] - \zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\mathsf{g}^{-1}(t,\vec{r})} \tag{III.27c}$$

with $\mathsf{D}(t,\vec{r})$ the strain rate tensor discussed in § II.1.3. Component-wise, this reads

$$\pi^{ij}(t,\vec{r}) \equiv -2\eta(t,\vec{r})\left[\mathsf{D}^{ij}(t,\vec{r}) - \frac{1}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]g^{ij}(t,\vec{r})\right] - \zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]g^{ij}(t,\vec{r}). \tag{III.27d}$$

In terms of the traceless rate-of-shear tensor (II.17b) or of its components (II.17d), one may alternatively write

$$\boldsymbol{\pi}(t,\vec{r}) \equiv -2\eta(t,\vec{r})\,\mathsf{S}(t,\vec{r}) - \zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\mathsf{g}^{-1}(t,\vec{r}) \tag{III.27e}$$

$$\pi^{ij}(t,\vec{r}) \equiv -2\eta(t,\vec{r})\,\mathsf{S}^{ij}(t,\vec{r}) - \zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]g^{ij}(t,\vec{r}). \tag{III.27f}$$

[(xxxvii)] *viskoser Spannungstensor*

This viscous stress tensor involves two novel characteristics of the medium, so-called *transport coefficients*, namely:

- the (dynamical) *shear viscosity*[(xxxviii)] η, which multiplies the traceless symmetric part of the velocity gradient tensor, i.e. the conveniently termed rate-of-shear tensor;

- the *bulk viscosity*, also called *second viscosity*,[(xxxix)] ζ, which multiplies the volume-expansion part of the velocity gradient tensor, i.e. the term proportional to $\vec{\nabla} \cdot \vec{v}(t, \vec{r})$.

The two corresponding contributions represent a *diffusive* transport of momentum in the fluid—representing a third type of transport besides the convective and thermal ones.

Remarks:

∗ In the case of a Newtonian fluid, the viscosity coefficients η and ζ are independent of the flow velocity. However, they still depend on the temperature and pressure of the fluid, so that they are not necessarily uniform and constant in a real flowing fluid.

∗ In an incompressible flow, $\vec{\nabla} \cdot \vec{v}(t, \vec{r}) = 0$, the last contribution to the momentum flux density (III.27) drops out. Thus, the bulk viscosity ζ only plays a role in compressible fluid motions.[(6)]

∗ Expression (III.27c) or (III.27d) of the viscous stress tensor assumes implicitly that the fluid is (locally) isotropic, since the coefficients η, ζ are independent of the directions i, j.

III.3.3 b Surface forces in a Newtonian fluid

The Cauchy stress tensor corresponding to the momentum flux density (III.27) of a Newtonian fluid is [cf. Eq. (III.23)]

$$\boldsymbol{\sigma}(t, \vec{r}) = -\mathcal{P}(t, \vec{r})\,\mathbf{g}^{-1}(t, \vec{r}) - \boldsymbol{\pi}(t, \vec{r}) \tag{III.28a}$$

that is, using the form (III.27e) of the viscous stress tensor

$$\boldsymbol{\sigma}(t, \vec{r}) = -\mathcal{P}(t, \vec{r})\,\mathbf{g}^{-1}(t, \vec{r}) + 2\eta(t, \vec{r})\,\mathbf{S}(t, \vec{r}) + \zeta(t, \vec{r})\big[\vec{\nabla} \cdot \vec{v}(t, \vec{r})\big]\mathbf{g}^{-1}(t, \vec{r}). \tag{III.28b}$$

Component-wise, this becomes

$$\sigma^{ij}(t, \vec{r}) = \left\{ -\mathcal{P}(t, \vec{r}) + \left[\zeta(t, \vec{r}) - \frac{2}{3}\eta(t, \vec{r})\right]\vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right\} g^{ij}(t, \vec{r}) + \eta(t, \vec{r})\left[\frac{\mathrm{d}v^i(t, \vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}v^j(t, \vec{r})}{\mathrm{d}x_i}\right]. \tag{III.28c}$$

Accordingly, the mechanical stress vector on an infinitesimally small surface element situated at point $\vec{r}$ with unit normal vector $\vec{e}_n(\vec{r})$ reads

$$\vec{T}_s(t, \vec{r}) = \boldsymbol{\sigma}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = \sum_{i,j=1}^{3} \left\{ \left[-\mathcal{P}(t, \vec{r}) + \left(\zeta(t, \vec{r}) - \frac{2}{3}\eta(t, \vec{r})\right)\vec{\nabla} \cdot \vec{v}(t, \vec{r})\right] g^i{}_j(t, \vec{r}) \right.$$
$$\left. + \eta(t, \vec{r})\left(\frac{\mathrm{d}v^i(t, \vec{r})}{\mathrm{d}x^j} + \frac{\mathrm{d}v_j(t, \vec{r})}{\mathrm{d}x_i}\right)\right\} n^j(\vec{r})\,\vec{e}_i(t, \vec{r}), \tag{III.29}$$

with $n^j(\vec{r})$ the coordinate of $\vec{e}_n(\vec{r})$ along direction j. One easily identifies the two components of this stress vector (cf. § I.4.1)

- the term proportional to $\sum g^i{}_j n^j \vec{e}_i = \vec{e}_n$ is the *normal stress* on the surface element. It consists of the usual hydrostatic pressure term $-\mathcal{P}\,\vec{e}_n$, and a second contribution, proportional to the local expansion rate $\vec{\nabla} \cdot \vec{v}$: in the compressible motion of a Newtonian—and more generally a dissipative—fluid, the normal stress is thus not only given by $-\mathcal{P}\,\vec{e}_n$, but includes additional contributions that vanish in the static case.

[(6)]As a consequence, the bulk viscosity is often hard to measure—one has to devise a compressible flow—so that it is actually not so well known for many substances, even well-studied ones [13].

[(xxxviii)]*Scherviskosität*   [(xxxix)]*Dehnviskosität, Volumenviskosität, zweite Viskosität*

- the remaining term is the *tangential stress*, proportional to the shear viscosity η. Accordingly, the value of the latter can be deduced from a measurement of the tangential force acting on a surface element, see § **??**.

As in § III.3.2 a, the external contact forces acting on a fluid element delimited by a surface $\mathcal{S}$ can easily be computed. Invoking the Stokes theorem yields

$$\oint_{\mathcal{S}} \vec{T}_s(t,\vec{r})\,\mathrm{d}^2\mathcal{S} = -\oint_{\mathcal{S}} \mathcal{P}(t,\vec{r})\,\vec{e}_n(\vec{r})\,\mathrm{d}^2\mathcal{S} - \oint_{\mathcal{S}} \boldsymbol{\pi}(t,\vec{r})\cdot\vec{e}_n(\vec{r})\,\mathrm{d}^2\mathcal{S}$$

$$= -\int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,\mathrm{d}^3\mathcal{V} + \int_{\mathcal{V}} \vec{\nabla}\cdot\boldsymbol{\pi}(t,\vec{r})\,\mathrm{d}^3\mathcal{V}$$

$$= -\int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,\mathrm{d}^3\mathcal{V} + \int_{\mathcal{V}} \vec{f}_{\mathrm{visc}}(t,\vec{r})\,\mathrm{d}^3\mathcal{V}, \tag{III.30a}$$

with the local viscous friction force density

$$\vec{f}_{\mathrm{visc}}(t,\vec{r}) = \sum_{i,j=1}^{3} \frac{\mathrm{d}}{\mathrm{d}x^i}\left\{\eta(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}^i(t,\vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}\mathsf{v}^j(t,\vec{r})}{\mathrm{d}x_i}\right]\right\}\vec{e}_j(t,\vec{r})$$

$$+ \vec{\nabla}\left\{\left[\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\vec{r})\right]\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right\}. \tag{III.30b}$$

Remark: Equations (III.16c) or equivalently (III.16d)—in the case of a perfect fluid— and Eqs. (III.28b)–(III.28c) for a Newtonian fluid, relating the Cauchy stress tensor to the flow velocity (and the pressure), are so-called *constitutive relations*. These actually define—together with expressions for the heat currents—the respective models.

III.3.3 c Navier–Stokes equation

Combining the viscous force (III.30b) with the generic equations (III.12), (III.14) and (III.15), the application of Newton's second law to a volume $\mathcal{V}$ of fluid leads to an identity between sums of volume integrals. Since this relation holds for any volume $\mathcal{V}$, it translates into an identity between the integrands, namely

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\mathrm{visc}}(t,\vec{r}) + \vec{f}_V(t,\vec{r}) \tag{III.31a}$$

or component-wise

$$\rho(t,\vec{r})\left\{\frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\mathsf{v}^i(t,\vec{r})\right\} = -\frac{\mathrm{d}\mathcal{P}(t,\vec{r})}{\mathrm{d}x_i} + \frac{\mathrm{d}}{\mathrm{d}x_i}\left\{\left[\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\vec{r})\right]\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right\}$$

$$+ \sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left\{\eta(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}^i(t,\vec{r})}{\mathrm{d}x_j} + \frac{\mathrm{d}\mathsf{v}^j(t,\vec{r})}{\mathrm{d}x_i}\right]\right\} + \left[\vec{f}_V(t,\vec{r})\right]^i \tag{III.31b}$$

for $i = 1, 2, 3$.

If the implicit dependence of the viscosity coefficients η and ζ on time and position is negligible, one may take them outside of the spatial derivatives. As a result, one obtains the (compressible) *Navier*[(i)]*–Stokes*[(j)] *equation*

$$\boxed{\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \eta\triangle\vec{\mathsf{v}}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right] + \vec{f}_V(t,\vec{r})}$$

$$\tag{III.32}$$

with $\triangle = \vec{\nabla}^2$ the Laplacian. This is a non-linear partial differential equation of second order, while the Euler equation (III.18) is of first order.

[(i)]C.-L. NAVIER, 1785–1836   [(j)]G. G. STOKES, 1819–1903

The difference between the order of the equations is not a mere detail: while the Euler equation looks like the limit $\eta, \zeta \to 0$ of the Navier–Stokes equation, this may not necessarily hold true for their solutions. This is for instance due to the fact that their respective boundary conditions differ.

In the case of an incompressible flow, the local expansion rate in the Navier–Stokes equation (III.32) vanishes, leading to the *incompressible Navier–Stokes equation*

$$\frac{\partial \vec{v}(t,\vec{r})}{\partial t} + \left[\vec{v}(t,\vec{r}) \cdot \vec{\nabla} \right] \vec{v}(t,\vec{r}) = -\frac{1}{\rho} \vec{\nabla} \mathcal{P}(t,\vec{r}) + \nu \triangle \vec{v}(t,\vec{r}), \tag{III.33}$$

with

$$\nu \equiv \eta/\rho \tag{III.34}$$

the *kinematic shear viscosity*.

Remark: The dimension of the dynamic viscosity coefficients η, ζ is $\mathsf{M}\,\mathsf{L}^{-1}\,\mathsf{T}^{-1}$ and the corresponding unit in the SI system is the Poiseuille[k], abbreviated Pa·s. In contrast, the kinematic viscosity has dimension $\mathsf{L}^2\,\mathsf{T}^{-1}$, i.e. depends only on space and time, hence its denomination.

III.3.3 d Boundary conditions

At the interface between a viscous fluid, in particular a Newtonian one, and another body—be it an obstacle in the flow, a wall containing the fluid, or even a second viscous fluid which is immiscible with the first one—the relative velocity between the fluid and the body must vanish. This holds not only for the normal component of the velocity ("impermeability" condition), as in perfect fluids, but also for the tangential one, to account for the friction forces. The latter requirement is often referred to as *no-slip condition*.

III.3.4 Higher-order dissipative fluid dynamics

Instead of considering only the first spatial derivatives of the velocity field in the momentum flux-density tensor (III.26), one may wish to also include the second derivatives, or even higher ones. Such assumptions lead to partial differential equations of motion, replacing the Navier–Stokes equation, of increasing order: *Burnett equation*, *super Burnett equation* [14, 15].

The domain of validity of such higher-order dissipative fluid models is a priori larger than that of first-order fluid dynamics, since it becomes possible to account for stronger gradients. On the other hand, this is at the cost of introducing a large number of new parameters besides the transport coefficients already present in Newtonian fluids. In parallel, the numerical implementation of the model becomes more involved, so that a macroscopic description does not necessarily represent the best approach.

III.4 Energy conservation, entropy balance

The conservation of mass and Newton's second law for linear momentum lead to four partial differential equations, one scalar—continuity equation (III.9)—and one vectorial—Euler (III.18) or Navier–Stokes (III.32)—, describing the coupled evolutions of five fields: $\rho(t,\vec{r})$, the three components of $\vec{v}(t,\vec{r})$ and $\mathcal{P}(t,\vec{r})$.[7] To fully determine the latter, a fifth equation is needed. For this last constraint, there are several possibilities.

A first alternative is if some of the *kinematic* properties of the fluid flow are imposed a priori. Thus, requiring that the motion should be steady or irrotational or incompressible... might suffice

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[7]The density of volume forces $\vec{f}_V$ or equivalently the corresponding potential energy per unit mass Φ, which stand for gravity or inertial forces, are given "from the outside" and not counted as a degree of freedom.

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[k]J.-L.-M. POISEUILLE, 1797–1869

to fully constrain the fluid flow for the geometry under consideration: we shall see several examples in the next three Chapters.

A second possibility, which will also be illustrated in Chap. **??**–**??**, is that of a *thermodynamic* constraint: isothermal flow, isentropic flow… For instance, one sees in thermodynamics that in an adiabatic process for an ideal gas, the pressure and volume of the latter obey the relation $\mathcal{P}\mathcal{V}^\gamma = $ constant, where γ denotes the ratio of the heat capacities at constant pressure ($C_\mathcal{P}$) and constant volume ($C_\mathcal{V}$). Since $\mathcal{V}$ is proportional to $1/\rho$, this so-called "adiabatic equation of state" provides the needed constraint relating pressure and mass density.

Eventually, one may argue that non-relativistic physics automatically implies a further conservation law besides those for mass and linear momentum, namely energy conservation. Thus, using the reasoning adopted in Secs. (III.2) and (III.3), the rate of change of the total energy—internal, kinetic and potential—of the matter inside a given volume equals the negative of the flow of energy through the surface delimiting this volume. In agreement with the first law of thermodynamics, one must take into account in the energy exchanged with the exterior of the volume not only the convective transport of internal, kinetic and potential energies, but also the exchange of the mechanical work of contact forces and—for dissipative fluids—of heat.

III.4.1 Energy and entropy conservation in perfect fluids

In non-dissipative non-relativistic fluids, energy is either transported convectively—as it accompanies some flowing mass of fluid—or exchanged in the form of the mechanical work of the pressure forces between neighboring regions. Mathematically, this is expressed at the local level by the equation

$$
\begin{aligned}
\frac{\partial}{\partial t}&\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{v}(t,\vec{r})^2 + e(t,\vec{r}) + \rho(t,\vec{r})\Phi(t,\vec{r})\right] \\
&+ \vec{\nabla}\cdot\left\{\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{v}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r}) + \rho(t,\vec{r})\Phi(t,\vec{r})\right]\vec{v}(t,\vec{r})\right\} = 0,
\end{aligned}
\tag{III.35}
$$

where e denotes the local density of internal energy and Φ the potential energy per unit mass of volume forces—assumed to be conservative—such that the acceleration $\vec{a}_V$ present in Eq. (III.19) equals $-\vec{\nabla}\Phi$.

Equation (III.35) will not be proven here—we shall see later in § **??** that it emerges as low-velocity limit of one of the equations of non-dissipative relativistic fluid dynamics. It is however clearly of the usual form for a conservation equation, involving

- the total energy density, consisting of the kinetic ($\frac{1}{2}\rho\vec{v}^2$), internal (e) and potential ($\rho\Phi$) energy densities; and

- the total energy flux density, which involves the previous three forms of energy, as well as that exchanged as mechanical work of the pressure forces.[8]

Remarks:

∗ The presence of pressure in the flux density, however not in the density, is reminiscent of the same property in definitions (III.21).

∗ The assumption that the volume forces are conservative is of course not innocuous. For instance, it does not hold for Coriolis forces, which means that one must be careful when working in a rotating reference frame.

---

[8]Remember that when a system with pressure $\mathcal{P}$ increases its volume by an amount $\mathrm{d}\mathcal{V}$, it exerts a mechanical work $\mathcal{P}\,\mathrm{d}\mathcal{V}$, "provided" to its exterior.

∗ The careful reader will have noticed that energy conservation (III.35) constitutes a fifth equation complementing the continuity and Euler equations (III.9) and (III.18), yet at the cost of introducing a new scalar field, the energy density, so that now a sixth equation is needed. The latter is provided by the thermal equation of state of the fluid, which relates its energy density, mass density and pressure.[9] In contrast to the other equations, this equation of state is not "dynamical", i.e. for instance it does not involve time or spatial derivatives, but is purely algebraic.

One can show—again, this will be done in the relativistic case (§ **??**), and can also be seen as special case of the result obtained for Newtonian fluids in § III.4.3—that in a perfect, non-dissipative fluid, the relation (III.35) expressing energy conservation locally, together with thermodynamic relations, lead to the local conservation of entropy, expressed as

$$\frac{\partial s(t,\vec{r})}{\partial t} + \vec{\nabla} \cdot \left[s(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r}) \right] = 0, \tag{III.36}$$

where $s(t,\vec{r})$ is the entropy density, while $s(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})$ represents the entropy flux density. The motion of a perfect fluid is thus automatically *isentropic*.

This equation, together with a thermodynamic relation, is sometimes more practical than the energy conservation equation (III.35), to which it is however totally equivalent.

III.4.2 Energy conservation in Newtonian fluids

In a real fluid, there exist further mechanisms for transporting energy besides the convective transport due to the fluid motion, namely *diffusion*, either of momentum or of energy:

- On the one hand, the viscous friction forces in the fluid, which already lead to the transport of momentum between neighboring fluid layers moving with different velocities, exert some work in the motion, which induces a diffusive transport of energy. This is accounted for by a contribution $\boldsymbol{\pi} \cdot \vec{\mathsf{v}}$ to the energy flux density—component-wise, a contribution $\sum_j \pi^i{}_j \mathsf{v}^j$ to the i-th component of the flux density—, with $\boldsymbol{\pi}$ the viscous stress tensor, given in the case of a Newtonian fluid by Eq. (III.27c).

- On the other hand, there is also *heat conduction* from the regions with higher temperatures towards those with lower temperatures. This transport is described by the introduction in the energy flux density of a heat current[(xl)] $\vec{\jmath}_Q(t,\vec{r}) = -\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})$—in accordance with the local formulation of *Fourier's law*,[(1)] see e.g. Sec. 1.2.1 in Ref. [2]—, with κ the *heat conductivity*[(xli)] of the fluid.

Taking into account these additional contributions, the local formulation of energy conservation in a Newtonian fluid reads

$$\begin{aligned}
\frac{\partial}{\partial t} &\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \rho(t,\vec{r})\Phi(t,\vec{r}) \right] \\
&+ \vec{\nabla} \cdot \Bigg\{ \left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathscr{P}(t,\vec{r}) + \rho(t,\vec{r})\Phi(t,\vec{r}) \right] \vec{\mathsf{v}}(t,\vec{r}) \\
&\qquad - \eta(t,\vec{r}) \left[\left(\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \right) \vec{\mathsf{v}}(t,\vec{r}) + \vec{\nabla}\left(\frac{\vec{\mathsf{v}}(t,\vec{r})^2}{2} \right) \right] \\
&\qquad - \left[\zeta(t,\vec{r}) - \frac{2\eta(t,\vec{r})}{3} \right] \left[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \right] \vec{\mathsf{v}}(t,\vec{r}) - \kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r}) \Bigg\} = 0.
\end{aligned} \tag{III.37}$$

[(9)]This is where the assumption of local thermodynamic equilibrium (§ I.1.3) plays a crucial role.

[(xl)] *Wärmestromvektor* [(xli)] *Wärmeleitfähigkeit*

[(1)]J. B. FOURIER, 1768–1830

with Φ the potential energy per unit mass due to conservative external volume forces. If the three transport coefficients η, ζ and κ vanish, this equation simplifies to that for perfect fluids, Eq. (III.35).

Remark: The energy flux density can be read off Eq. (III.37), since it represents the term between curly brackets. Dropping the external potential Φ, one can check that it can also be written as

$$\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r}) - 2\eta(t,\vec{r})\,\mathsf{S}(t,\vec{r})\cdot\vec{\mathrm{v}}(t,\vec{r})$$
$$- \zeta(t,\vec{r})\left[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r}) - \kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r}), \quad \text{(III.38)}$$

with $\mathsf{S}(t,\vec{r})$ the traceless symmetric rate-of-shear tensor. One recognizes the various physical sources of energy transport.

III.4.3 Entropy balance in Newtonian fluids

In a real fluid, with viscous friction forces and heat conductivity, one can expect a priori that the transformation of mechanical energy into heat will lead in general to an increase in entropy, at least as long as one considers a closed and isolated system.

Consider a volume $\mathcal{V}$ of flowing Newtonian fluid, delimited by a surface $\mathcal{S}$ at each point $\vec{r}$ of which the boundary conditions

$$\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\mathrm{e}}_n(\vec{r}) = 0 \quad \text{and} \quad \vec{\jmath}_Q(t,\vec{r})\cdot\vec{\mathrm{e}}_n(\vec{r}) = 0$$

hold, where $\vec{\mathrm{e}}_n(\vec{r})$ denotes the unit normal vector to $\mathcal{S}$ at $\vec{r}$. Physically, these boundary conditions mean than neither matter nor heat flows across the surface $\mathcal{S}$, so that the system inside $\mathcal{S}$ is closed and isolated. To completely exclude energy exchanges with the exterior of $\mathcal{S}$, it is also assumed that there are no external volume forces acting on the fluid inside volume $\mathcal{V}$. We shall investigate the implications of the continuity equation (III.10), the Navier–Stokes equation (III.32), and the energy conservation equation (III.37) for the total entropy S of the fluid inside $\mathcal{V}$. For the sake of brevity, the variables t, $\vec{r}$ will be omitted in the remainder of this Section.

Starting with the energy conservation equation (III.37), the contribution

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\,\vec{\mathrm{v}}^2\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\,\vec{\mathrm{v}}^2\right)\vec{\mathrm{v}}\right]$$

in its first two lines can be replaced by

$$\rho\,\vec{\mathrm{v}}\cdot\frac{\partial\vec{\mathrm{v}}}{\partial t} + \frac{1}{2}\frac{\partial\rho}{\partial t}\,\vec{\mathrm{v}}^2 + \frac{1}{2}\left[\vec{\nabla}\cdot\left(\rho\,\vec{\mathrm{v}}\right)\right]\vec{\mathrm{v}}^2 + \sum_{i=1}^{3}\rho\mathrm{v}_i(\vec{\mathrm{v}}\cdot\vec{\nabla})\mathrm{v}^i = \sum_{i=1}^{3}\rho\mathrm{v}_i\left[\frac{\partial\mathrm{v}^i}{\partial t} + \left(\vec{\mathrm{v}}\cdot\vec{\nabla}\right)\mathrm{v}^i\right], \quad \text{(III.39a)}$$

where the continuity equation (III.9) was used.

As recalled in Appendix **??**, the fundamental thermodynamic relation $U = TS - \mathcal{P}V + \mu N$ gives on the one hand $e + \mathcal{P} = Ts + \mu n$, which leads to

$$\vec{\nabla}\cdot\left[(e+\mathcal{P})\vec{\mathrm{v}}\right] = T\vec{\nabla}\cdot\left(s\vec{\mathrm{v}}\right) + \mu\vec{\nabla}\cdot\left(n\vec{\mathrm{v}}\right) + \vec{\mathrm{v}}\cdot\left(s\vec{\nabla}T + n\vec{\nabla}\mu\right) = T\vec{\nabla}\cdot\left(s\vec{\mathrm{v}}\right) + \mu\vec{\nabla}\cdot\left(n\vec{\mathrm{v}}\right) + \vec{\mathrm{v}}\cdot\vec{\nabla}\mathcal{P}, \quad \text{(III.39b)}$$

where the second identity follows from the Gibbs–Duhem relation $\mathrm{d}\mathcal{P} = s\,\mathrm{d}T + n\,\mathrm{d}\mu$. On the other hand, it leads to $\mathrm{d}e = T\,\mathrm{d}s + \mu\,\mathrm{d}n$, which under consideration of the continuity equation for particle number yields

$$\frac{\partial e}{\partial t} = T\frac{\partial s}{\partial t} + \mu\frac{\partial n}{\partial t} = T\frac{\partial s}{\partial t} - \mu\vec{\nabla}\cdot\left(n\vec{\mathrm{v}}\right). \quad \text{(III.39c)}$$

With the help of relations (III.39a)–(III.39c), the energy conservation equation (III.37) can be rewritten as

$$\sum_{i=1}^{3} \rho \mathsf{v}_i \left[\frac{\partial \mathsf{v}^i}{\partial t} + (\vec{\mathsf{v}} \cdot \vec{\nabla}) \mathsf{v}^i \right] + T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s\vec{\mathsf{v}}) + \vec{\mathsf{v}} \cdot \vec{\nabla} \mathscr{P} =$$

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x^j} \left[\eta \left(\frac{\partial \mathsf{v}^i}{\partial x_j} + \frac{\partial \mathsf{v}^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{\mathsf{v}} \right) \mathsf{v}_i \right] + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[\zeta (\vec{\nabla} \cdot \vec{\mathsf{v}}) \mathsf{v}_i \right] + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \quad \text{(III.39d)}$$

Multiplying the i-th component of Eq. (III.31b) by v_i gives

$$\rho \mathsf{v}_i \left[\frac{\partial \mathsf{v}^i}{\partial t} + (\vec{\mathsf{v}} \cdot \vec{\nabla}) \mathsf{v}^i \right] + \mathsf{v}_i \frac{\partial \mathscr{P}}{\partial x_i} = \sum_{j=1}^{3} \mathsf{v}_i \frac{\partial}{\partial x^j} \left[\eta \left(\frac{\partial \mathsf{v}^i}{\partial x_j} + \frac{\partial \mathsf{v}^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{\mathsf{v}} \right) \right] + \mathsf{v}_i \frac{\partial}{\partial x^i} (\zeta \vec{\nabla} \cdot \vec{\mathsf{v}}).$$

Subtracting this identity, summed over $i = 1, 2, 3$, from Eq. (III.39d), yields

$$T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s\vec{\mathsf{v}}) = \eta \sum_{i,j=1}^{3} \frac{\partial \mathsf{v}_i}{\partial x^j} \left(\frac{\partial \mathsf{v}^i}{\partial x_j} + \frac{\partial \mathsf{v}^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) + \zeta (\vec{\nabla} \cdot \vec{\mathsf{v}})^2 + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \quad \text{(III.40)}$$

On the right hand side of this equation, one may use the identity

$$\frac{1}{2} \sum_{i,j=1}^{3} \left(\frac{\partial \mathsf{v}^i}{\partial x_j} + \frac{\partial \mathsf{v}^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left(\frac{\partial \mathsf{v}_i}{\partial x^j} + \frac{\partial \mathsf{v}_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{v} \right) = \sum_{i,j=1}^{3} \left(\frac{\partial \mathsf{v}^i}{\partial x_j} + \frac{\partial \mathsf{v}^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \frac{\partial \mathsf{v}_j}{\partial x^i}, \quad \text{(III.41a)}$$

which follows from the fact that both symmetric terms $\partial \mathsf{v}_i / \partial x^j$ and $\partial \mathsf{v}_j / \partial x^i$ from the left member give the same contribution, while the term in g_{ij} yields a zero contribution, since it multiplies a traceless term.
Additionally, one has

$$\vec{\nabla} \cdot (\kappa \vec{\nabla} T) = T \vec{\nabla} \cdot \left(\frac{\kappa \vec{\nabla} T}{T} \right) + \frac{\kappa}{T} (\vec{\nabla} T)^2. \quad \text{(III.41b)}$$

All in all, Eqs. (III.40) and (III.41) lead to

$$\frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s\vec{\mathsf{v}}) - \vec{\nabla} \cdot \left(\frac{\kappa \vec{\nabla} T}{T} \right) = \frac{\eta}{2T} \sum_{i,j=1}^{3} \left(\frac{\partial \mathsf{v}^i}{\partial x_j} + \frac{\partial \mathsf{v}^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left(\frac{\partial \mathsf{v}_i}{\partial x^j} + \frac{\partial \mathsf{v}_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{\mathsf{v}} \right)$$

$$+ \frac{\zeta}{T} (\vec{\nabla} \cdot \vec{\mathsf{v}})^2 + \kappa \frac{(\vec{\nabla} T)^2}{T^2}. \quad \text{(III.42a)}$$

This may still be recast in the slightly more compact form

$$\boxed{\begin{aligned} &\frac{\partial s(t,\vec{r})}{\partial t} + \vec{\nabla} \cdot \left[s(t,\vec{r}) \, \vec{\mathsf{v}}(t,\vec{r}) - \kappa(t,\vec{r}) \frac{\vec{\nabla} T(t,\vec{r})}{T(t,\vec{r})} \right] = \\ &\frac{1}{T(t,\vec{r})} \left\{ 2\eta(t,\vec{r}) \, \mathsf{S}(t,\vec{r}) : \mathsf{S}(t,\vec{r}) + \zeta(t,\vec{r}) \left[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \right]^2 + \kappa(t,\vec{r}) \frac{\left[\vec{\nabla} T(t,\vec{r}) \right]^2}{T(t,\vec{r})} \right\} \end{aligned}}$$

$$\text{(III.42b)}$$

with $\mathsf{S} : \mathsf{S} \equiv \mathsf{S}_{ij} \mathsf{S}^{ij}$ the scalar obtained by doubly contracting the rate-of-shear tensor with itself.

This equation can then be integrated over the $\mathcal{V}$ occupied by the fluid:

- When computing the integral of the divergence term on the left hand side with the Stokes theorem, it vanishes thanks to the boundary conditions imposed at the surface $\mathcal{S}$;

- the remaining term in the left member is simply the time derivative $\mathrm{d}S/\mathrm{d}t$ of the total entropy of the closed system;

- if all three transport coefficients η, ζ, κ are *positive*, then it is also the case of the three terms on the right hand side.

One thus finds $\dfrac{\mathrm{d}S}{\mathrm{d}t} \geq 0$, in agreement with the second law of thermodynamics.

Remarks:

∗ The previous derivation may be seen as a proof that the transport coefficients must be positive to ensure that the second law of thermodynamics holds.

∗ If all three transport coefficients η, ζ, κ vanish, i.e. in the case of a non-dissipative fluid, Eq. (III.42) simply reduces to the entropy conservation equation in perfect fluids (III.36).

Bibliography for Chapter III

- Feynman [10, 11] Chapter 40–2 & 41–1, 41–2.

- Guyon *et al.* [2] Chapters 3.3, 4.1–4.3, 5.1, 5.2.

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- Fließbach [16] Chapter 32.