III.3 Momentum balance: Euler and Navier–Stokes equations

For a closed system Σ with total linear momentum $\vec{P}$ with respect to a given reference frame $\mathcal{R}$, Newton's second law reads

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \vec{F}(t), \tag{III.11}$$

with $\vec{F}$ the sum of the "external" forces acting on Σ.

    The left hand side of this equation can be transformed with the help of Reynolds' transport theorem (III.5), irrespective of any assumption on the fluid under consideration (§ III.3.1). In contrast, the forces acting on a fluid element, more precisely the forces exerted by the neighboring elements, do depend on the (assumed) properties of the fluid. The two most widespread models used for fluids are that of a perfect fluid, which leads to the Euler equation (§ III.3.2), and of a Newtonian fluid, for which Newton's second law (III.11) translates into the Navier–Stokes equation (§ III.3.3).

    Throughout this Section, we use the shorter designation "momentum" instead of the more accurate "linear momentum".

III.3.1 Material derivative of momentum

    As already noted shortly below Eq. (III.1), the specific density associated with the momentum $\vec{P}(t)$ is simply the flow velocity $\vec{\mathrm{v}}(t,\vec{r})$. Applying Reynolds' theorem (III.5) to the momentum of the material system contained at time t inside a control volume $\mathcal{V}$, the material derivative on the left hand side of Newton's law (III.11) can be recast as

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} \vec{\mathrm{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\mathrm{d}^3\vec{r}\right] + \oint_{\partial\mathcal{V}} \vec{\mathrm{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\cdot\mathrm{d}^2\vec{\mathcal{S}}. \tag{III.12}$$

    Both terms on the right hand side can be transformed to yield more tractable expressions. On the one hand, since the volume $\mathcal{V}$ is motionless in the reference frame $\mathcal{R}$, the time derivative can

be taken inside the integral. Its action on $\rho(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})$ is then given by the usual product rule. On the other hand, one can show the identity

$$\oint_{\partial\mathcal{V}} \vec{\mathbf{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})\cdot\mathrm{d}^2\vec{\mathcal{S}} = \int_{\mathcal{V}}\left\{-\vec{\mathbf{v}}(t,\vec{r})\,\frac{\partial\rho(t,\vec{r})}{\partial t} + \rho(t,\vec{r})\big[\vec{\mathbf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathbf{v}}(t,\vec{r})\right\}\mathrm{d}^3\vec{r}. \qquad \text{(III.13)}$$

All in all, one thus obtains

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \int_{\mathcal{V}}\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathbf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathbf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathbf{v}}(t,\vec{r})\right\}\mathrm{d}^3\vec{r} = \int_{\mathcal{V}}\rho(t,\vec{r})\,\frac{\mathrm{D}\vec{\mathbf{v}}(t,\vec{r})}{\mathrm{D}t}\,\mathrm{d}^3\vec{r}. \qquad \text{(III.14)}$$

Proof of relation (III.13): consider the i-th component of the vector defined by the surface integral on the left hand side of that identity. Gauss's divergence theorem gives

$$\oint_{\partial\mathcal{V}}\big[\mathbf{v}^i(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})\big]\cdot\mathrm{d}^2\vec{\mathcal{S}} = \int_{\mathcal{V}}\vec{\nabla}\cdot\big[\mathbf{v}^i(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})\big]\,\mathrm{d}^3\vec{r}.$$

The action of the differential operator yields $\mathbf{v}^i(t,\vec{r})\,\vec{\nabla}\cdot\big[\rho(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})\big] + \rho(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})\cdot\vec{\nabla}\mathbf{v}^i(t,\vec{r})$: the divergence in the first term can be expressed according to the continuity equation (III.9) as the negative of the time derivative of the mass density, leading to

$$\vec{\nabla}\cdot\big[\mathbf{v}^i(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})\big] = -\mathbf{v}^i(t,\vec{r})\,\frac{\partial\rho(t,\vec{r})}{\partial t} + \rho(t,\vec{r})\big[\vec{\mathbf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\mathbf{v}^i(t,\vec{r}).$$

This relation holds for all three components, from where Eq. (III.13) follows. $\qquad\square$

Remark: The derivation of Eq. (III.14) relies on purely algebraic transformations, either as encoded in Reynolds' transport theorem, or when going from relation (III.12) to (III.14). That is, it does not imply any model—apart from that of a continuous medium—for the fluid properties. In particular, whether or not dissipative effects are important in the fluid did not play any role here.

III.3.2 Perfect fluid: Euler equation

In this section, we first introduce the notion, or rather the model, of a *perfect fluid*, which is defined by the choice of a specific ansatz for the stress tensor that encodes the contact forces between neighboring fluid elements (§ III.3.2 a). Using that ansatz and the results of the previous paragraph, Newton's second law (III.11) is shown in § III.3.2 b to be equivalent to a local formulation, the so-called *Euler equation*. Eventually, the latter is recast in the generic form for a local conservation or balance equation, involving the time derivative of a local density and the divergence of the corresponding flux density (§ III.3.2 e).

III.3.2 a Forces in a perfect fluid

The forces in a fluid were already discussed on a general level in § I.4.1. Thus, the total force on the right hand side of Eq. (III.11) consists of volume and surface forces, which can respectively be expressed as a volume or a surface integral

$$\vec{F}(t) = \int_{\mathcal{V}}\vec{f}_V(t,\vec{r})\,\mathrm{d}^3\vec{r} + \oint_{\mathcal{S}}\vec{T}_s(t,\vec{r})\,\mathrm{d}^2\mathcal{S}, \qquad \text{(III.15)}$$

where $\vec{f}_V$ denotes the local density of body forces, while $\vec{T}_s$ is the mechanical stress vector introduced in Eq. (I.20). The latter will now allow us to introduce various models of fluids.

The first, simplest model is that of a *perfect fluid*, or *ideal fluid*:

> A perfect fluid *is a fluid in which there are no shear stresses nor heat conduction.*   (III.16a)

Stated differently, at every point of a perfect fluid the stress vector $\vec{T}_s$ on a (test) surface element $\mathrm{d}^2\mathcal{S}$ moving with the fluid is normal to $\mathrm{d}^2\mathcal{S}$, irrespective of whether the fluid is at rest or in motion.

That is, introducing the normal unit vector $\vec{e}_n(\vec{r})$ to $\mathrm{d}^2\mathcal{S}$ oriented towards the exterior of the material region acted upon,[3] one may write

$$\vec{T}_s(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\vec{e}_n(\vec{r}), \tag{III.16b}$$

with $\mathcal{P}(t,\vec{r})$ the pressure at position $\vec{r}$. Accordingly, the mechanical stress tensor in a perfect fluid in a reference frame $\mathcal{R}$ which is moving with the fluid is given by

$$\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\mathbf{g}^{-1}(t,\vec{r}), \tag{III.16c}$$

with $\mathbf{g}^{-1}$ the inverse metric tensor, just like in a fluid at rest [Eq. (I.23)]. In a given coordinate system in $\mathcal{R}$, the $\binom{2}{0}$-components of $\boldsymbol{\sigma}$ thus simply read

$$\boldsymbol{\sigma}^{ij}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,g^{ij}(t,\vec{r}) \tag{III.16d}$$

i.e. the $\binom{1}{1}$-components are $\boldsymbol{\sigma}^i{}_j(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\,\delta^i{}_j$.

Using relation (III.16b), the total surface forces in Eq. (III.15) can be transformed into a volume integral:

$$\oint_{\mathcal{S}} \vec{T}_s(t,\vec{r})\,\mathrm{d}^2\mathcal{S} = -\oint_{\mathcal{S}} \mathcal{P}(t,\vec{r})\,\vec{e}_n(\vec{r})\,\mathrm{d}^2\mathcal{S} = -\oint_{\mathcal{S}} \mathcal{P}(t,\vec{r})\,\mathrm{d}^2\vec{\mathcal{S}} = -\int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,\mathrm{d}^3\vec{r}, \tag{III.17}$$

where the last identity follows from a corollary of the usual divergence theorem.

Remark: Although this might not be intuitive at first, the pressure $\mathcal{P}(t,\vec{r})$ entering Eqs. (III.16b)–(III.16d) is actually the hydrostatic pressure already introduced in the definition of the mechanical stress in a fluid at rest, see Eq. (I.23). One heuristic justification is that the stresses are defined as the forces per unit area exerted by a piece of fluid situated on one side of a surface on the fluid situated on the other side. Even if the fluid is moving, the two fluid elements on both sides of the surface—as well as the comoving test surface—have the same velocity,[4] i.e. their relative velocity vanishes, just like in a fluid at rest.

III.3.2 b Euler equation

Gathering Eqs. (III.11), (III.14), (III.15) and (III.17) yields

$$\int_{\mathcal{V}} \rho(t,\vec{r})\left\{\frac{\partial\vec{v}(t,\vec{r})}{\partial t} + \left[\vec{v}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{v}(t,\vec{r})\right\}\mathrm{d}^3\vec{r} = \int_{\mathcal{V}}\left[-\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_V(t,\vec{r})\right]\mathrm{d}^3\vec{r}.$$

Since this identity must hold irrespective of the control volume $\mathcal{V}$, the integrands on both sides must be equal. That is, the various fields they involve obey the *Euler equation*

$$\boxed{\rho(t,\vec{r})\left\{\frac{\partial\vec{v}(t,\vec{r})}{\partial t} + \left[\vec{v}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{v}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_V(t,\vec{r}).} \tag{III.18}$$

Remarks:

∗ The term in curly brackets on the left hand side is exactly the acceleration (I.17) of a material point, as in Newton's second law.

∗ Due to the convective term $(\vec{v}\cdot\vec{\nabla})\vec{v}$, the Euler equation is a *nonlinear* partial differential equation.

---

[3]Cf. the discussion between Eqs. (I.21a)–(I.21c).
[4]... thanks to the usual continuity assumption: this no longer holds at a discontinuity!

$*$ Besides Newton's second law for linear momentum, one could also think of investigating the consequence of its analogue for angular momentum. Since we have assumed that the material points do not have any intrinsic spin, the conservation of angular momentum, apart from leading to the necessary symmetry of the stress tensor—which is realized in a perfect fluid, see Eq. (III.16c) or (III.16d), and will also hold in a Newtonian fluid, see Eq. (III.27)—does not bring any new dynamical equation.

III.3.2 c Boundary conditions

To fully formulate the mathematical problem representing a given fluid flow, one must also specify *boundary conditions* for the various partial differential equations. These conditions reflect the geometry of the problem under consideration.

- Far from an obstacle or from walls, one may specify a given pattern for the flow velocity field. For instance, one may require that the flow be uniform "at infinity", as e.g. for the motion far from the rotating cylinder in Fig. **??** illustrating the geometry of the Magnus effect.

- At an obstacle, in particular at a wall, the component of velocity *perpendicular* to the obstacle should vanish: that is, the fluid cannot penetrate the obstacle or wall, which makes sense and will be hereafter often referred to as *impermeability* condition. In case the obstacle is itself in motion, one should consider the (normal component of the) *relative* velocity of the fluid with respect to the obstacle.

  On the other hand, the model of a perfect fluid, in which there is by definition no friction, does not specify the value of the tangential component of the fluid relative velocity at an obstacle.

III.3.2 d Alternative forms of the Euler equation

In practice, the volume forces acting on a fluid element are often proportional to its mass, as are e.g. the gravitational, Coriolis or centrifugal forces. Therefore, it is rather natural to introduce the corresponding force density per unit mass, instead of per unit volume:

$$\vec{a}_V(t,\vec{r}) \equiv \frac{\mathrm{d}\vec{F}_V(t,\vec{r})}{\mathrm{d}M(t,\vec{r})} = \frac{\vec{f}_V(t,\vec{r})}{\rho(t,\vec{r})}.$$

With the help of this "specific density of body forces", which has the dimension of an acceleration, the Euler equation (III.18) can be recast as

$$\frac{\mathrm{D}\vec{v}(t,\vec{r})}{\mathrm{D}t} = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{a}_V(t,\vec{r}). \qquad (\text{III.19})$$

The interpretation of this form is quite straightforward: the acceleration of a material point (left hand side) is the sum of the acceleration due to the pressure forces and the acceleration due to volume forces (right hand side).

Alternatively, one may use the identity (in which the time and position variables have been omitted for the sake of brevity)

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}\left(\frac{\vec{v}^2}{2}\right) - (\vec{v}\cdot\vec{\nabla})\vec{v},$$

which can be proved either starting from the usual formula for the double cross product—with a small twist when applying the differential operator to a vector squared—or by working component by component. Recognizing in the rightmost term the convective part of the Euler equation, one can rewrite the latter, or equivalently Eq. (III.19), as

$$\frac{\partial \vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{\mathsf{v}}(t,\vec{r})^2}{2}\right] - \vec{\mathsf{v}}(t,\vec{r}) \times \vec{\omega}(t,\vec{r}) = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{a}_V(t,\vec{r}),$$

(III.20)

where we have made use of the vorticity vector (II.11). Note that the second term on the left hand side of this equation involves the (gradient of the) kinetic energy per unit mass dK/dM.

In § **??**, we shall see yet another form of the Euler equation [Eq. (**??**)], involving thermodynamic functions other than the pressure.

III.3.2 e The Euler equation as a balance equation

The Euler equation can be rewritten in the generic form for a balance equation, namely as the identity of the sum of the time derivative of a density and the divergence of a flux density with a source term—which vanishes if the quantity under consideration is conserved. Accordingly, we first introduce two

Definitions: One associates with the i-th component in a given coordinate system of the momentum of a material system its

- *density*[(xxxv)] $\rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})$ and   (III.21a)

- *flux density*[(xxxvi)] (in direction j) $\mathsf{T}^{ij}(t,\vec{r}) \equiv \rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r}) + \mathcal{P}(t,\vec{r})\,g^{ij}(t,\vec{r})$,   (III.21b)

with g^{ij} the components of the inverse metric tensor $\mathbf{g}^{-1}$.

Physically, T^{ij} represents the amount of momentum along $\vec{e}_i$ transported per unit time through a unit surface[(5)] perpendicular to the direction of $\vec{e}_j$—i.e. transported in direction j. That is, it is the i-th component of the force upon a test unit surface with normal unit vector $\vec{e}_j$.

The first contribution to T^{ij}—namely the transported momentum density multiplied by the velocity—arises from the convective transport represented by the macroscopic motion. On the other hand, the second term, involving pressure, represents the momentum transport due to the thermal, random motion of the atoms of the fluid.

Remarks:

∗ As thermal motion is random and (statistically) isotropic, it does not contribute to the momentum density $\rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r})$, only to the momentum flux density.

∗ In tensor notation, the momentum flux density (III.21b), viewed as a $\binom{2}{0}$-tensor, is given by

$$\mathbf{T}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r}) \otimes \vec{\mathsf{v}}(t,\vec{r}) + \mathcal{P}(t,\vec{r})\mathbf{g}^{-1}(t,\vec{r}) \quad \text{for a perfect fluid.}$$

(III.22)

∗ Given its physical meaning, the momentum flux (density) tensor $\mathbf{T}$ is obviously related to the Cauchy stress tensor $\boldsymbol{\sigma}$. More precisely, $\mathbf{T}$ represents the forces exerted by a material point on its neighbors, as measured in an arbitrary reference frame, in which the fluid velocity is $\vec{\mathsf{v}}(t,\vec{r})$. In turn $\boldsymbol{\sigma}$ stands for the stresses acting upon the material point due to its neighbors, as measured in the reference frame in which the material point is at rest, such that $\vec{\mathsf{v}}(t,\vec{r}) = \vec{0}$. In the latter reference frame, invoking Newton's third law—which in continuum mechanics is referred to as *Cauchy's fundamental lemma*—, the two tensors are simply opposite to each other. Coming back to an arbitrary reference frame, one may write

$$\mathbf{T}(t,\vec{r}) = \rho(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r}) \otimes \vec{\mathsf{v}}(t,\vec{r}) - \boldsymbol{\sigma}(t,\vec{r})$$

(III.23)

where $\boldsymbol{\sigma}(t,\vec{r})$ is given by Eq. (III.16c) for a perfect fluid.

[(5)]... which must be immobile in the reference frame in which the fluid has the velocity $\vec{\mathsf{v}}$ entering definition (III.21b).

[(xxxv)] *Impulsdichte*   [(xxxvi)] *Impulsstromdichte*

∗ Building on the previous remark, the absence of shear stress defining a perfect fluid can be reformulated as a condition of the momentum flux tensor:

> *A perfect fluid is a fluid at each point of which one can find a local velocity, such that for an observer moving with that velocity the fluid is locally isotropic. The momentum flux tensor is thus diagonal in the observer's reference frame.* (III.24)

The observer traveling with the same velocity as the local fluid velocity will be hereafter referred to as *comoving observer*, and the reference frame in which she is sitting as *local rest frame*.

Using definitions (III.21), one easily checks that the Euler equation (III.18) is equivalent to the balance equations (for $j = 1, 2, 3$)

$$\frac{\partial}{\partial t}\big[\rho(t, \vec{r})\, \mathsf{v}^j(t, \vec{r})\big] + \sum_{i=1}^{3} \frac{\mathrm{d}\mathsf{T}^{ij}(t, \vec{r})}{\mathrm{d}x^i} = f_V^j(t, \vec{r}). \qquad \text{(III.25a)}$$

with f_V^j the j-th component of the volume force density and $\mathrm{d}/\mathrm{d}x^i$ the covariant derivatives (see Appendix **??**), that coincide with the partial derivatives in Cartesian coordinates.

Proof: For the sake of brevity, the $(t, \vec{r})$-dependence of the various fields will not be specified. With the product rule and the definition of T^{ij}, one finds

$$\frac{\partial(\rho\mathsf{v}^j)}{\partial t} + \sum_{i=1}^{3} \frac{\mathrm{d}\mathsf{T}^{ij}}{\mathrm{d}x^i} = \frac{\partial\rho}{\partial t}\mathsf{v}^j + \rho\frac{\partial\mathsf{v}^j}{\partial t} + \sum_{i=1}^{3}\mathsf{v}^j\frac{\mathrm{d}(\rho\mathsf{v}^i)}{\mathrm{d}x^i} + \sum_{i=1}^{3}\rho\mathsf{v}^i\frac{\mathrm{d}\mathsf{v}^j}{\mathrm{d}x^i} + \sum_{i=1}^{3}g^{ij}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x^i}$$

$$= \mathsf{v}^j\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{\mathsf{v}})\right] + \rho\left[\frac{\partial\mathsf{v}^j}{\partial t} + (\vec{\mathsf{v}}\cdot\vec{\nabla})\mathsf{v}^j\right] + \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x_j},$$

where we have used $\sum_i g^{ij}\mathrm{d}/\mathrm{d}x^j = \mathrm{d}/\mathrm{d}x_j$. The first term between square brackets vanishes thanks to the continuity equation (III.9). In turn, the second term is precisely the j-th component of the left member of the Euler equation (III.18), i.e. it equals the j-th component of $\vec{f}_V$ minus the third term, which represents the j-th component of $\vec{\nabla}\mathcal{P}$. □

In tensor notation, Eq. (III.25a) reads

$$\frac{\partial}{\partial t}\big[\rho(t, \vec{r})\, \vec{\mathsf{v}}(t, \vec{r})\big] + \vec{\nabla}\cdot\mathsf{T}(t, \vec{r}) = \vec{f}_V(t, \vec{r}), \qquad \text{(III.25b)}$$

where the action of the divergence on a $\binom{2}{0}$-tensor is defined through its components, which is to be read off Eq. (III.25a).