Chapter III

Fundamental equations of non-relativistic fluid dynamics

Some of the most fundamental laws of physics are conservation equations for various quantities: energy, linear and angular momentum, electric charge, and so on. When applying these laws to many-body systems, in particular to continuous media like moving fluids, care must be taken to consider isolated and closed systems, to ensure their validity. At the very least, the amount of quantity exchanged with the exterior of the system—for example the change in linear momentum per unit time due to external forces, as given by Newton's second law, or the change in energy due to the mechanical work of these forces—must be quantifiable.

When this is the case, it is possible to re-express global conservation laws or more generally balance equations—given in terms of macroscopic quantities like total mass, total energy, total momentum, etc.—in a local form involving densities, using the generic recipe provided by Reynolds'[(h)] transport theorem (Sec. III.1). In the framework of a non-relativistic theory, in which the mass or equivalently the particle number of a closed system is conserved, one may thus derive a general continuity equation, holding at every point of the fluid (Sec. III.2).

The same approach may be followed to derive equations expressing the time evolution of momentum or energy under the influence of external forces acting at every point of the fluid. In either case, it is however necessary to account for the possibility that several physical phenomena may contribute to the transport of momentum and energy. One defines various fluid-dynamical models according to the level of approximation at which the possible forms of transport are incorporated in the description. To those different models correspond distinct sets of equations for the local expression of Newton's second law (Sec. III.3) or of energy balance (Sec. III.4).

[(h)] O. Reynolds, 1842–1912

III.1 Reynolds transport theorem

The material derivative of a quantity was already introduced in § I.3.4, where its action on a local function of both time t and position vector $\vec{r}$ was defined. In this Section, we shall derive a formula for the substantial derivative of an extensive physical quantity carried by a "macroscopic" material system. This formula will in the remainder of the Chapter represent the key relation that will allow us to express the usual (conservation) laws of Newtonian mechanics, which hold for closed systems, in terms of Eulerian variables.

III.1.1 Closed system, open system

Consider the motion of a continuous medium, in particular a flowing fluid, described in a reference frame $\mathcal{R}$. Let $\mathcal{S}$ be an arbitrary closed geometrical surface that remains fixed in $\mathcal{R}$. This surface will hereafter be referred to as *control surface*, and the geometrical volume $\mathcal{V}$ it encloses as *control volume*. Due to the macroscopic transport of matter in the flowing medium, the fluid contained inside a given control surface represents an *open system*, which can exchange matter and energy with its exterior as time elapses.

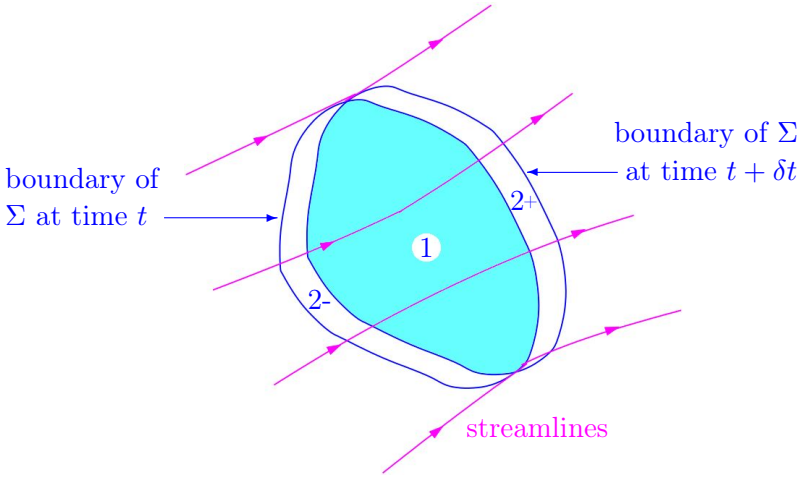


Figure III.1 – Time evolution of a closed material system transported in the motion of a continuous medium.

In contrast, let Σ be a *closed* system, whose constituent material points occupy at some given time t the volume $\mathcal{V}_\Sigma(t)$. At a shortly later time $t+\delta t$, the material system Σ has moved and now occupies a new volume $\mathcal{V}_\Sigma(t+\delta t)$. On Fig. III.1, one can distinguish between three regions in position space:

- (1), which is common to the successive positions of Σ at t and $t+\delta t$; the corresponding volume will be denoted $\mathcal{V}$ and its surface $\partial\mathcal{V}$;
- (2−), which is left behind by Σ between t and $t+\delta t$;
- (2+), into which Σ penetrates between t and $t+\delta t$.

Note already that in the limit δt going to zero, the volumes $\mathcal{V}_\Sigma(t)$, $\mathcal{V}_\Sigma(t+\delta t)$ and $\mathcal{V}$ coincide; more precisely, their difference is of order δt.

III.1.2 Material derivative of an extensive quantity

Let $\mathcal{G}(t)$ be one of the extensive quantities that characterize a macroscopic physical property of the closed material system Σ. To this extensive quantity, one can associate at every point $\vec{r}$ the corresponding intensive *specific density* $g_{\rm m}(t,\vec{r})$, defined as the local amount of $\mathcal{G}$ per unit mass of

matter. Denoting by $\mathrm{d}\mathcal{G}(t,\vec{r})$ resp. $\mathrm{d}M(t,\vec{r})$ the amount of $\mathcal{G}$ resp. the mass inside a small material volume at position $\vec{r}$ at time t, one can write symbolically

$$g_{\mathrm{m}}(t,\vec{r}) = \frac{\mathrm{d}\mathcal{G}(t,\vec{r})}{\mathrm{d}M(t,\vec{r})}, \tag{III.1}$$

where the notation with differentials is used to suggest that the identity holds in the limit of a small material volume.

For instance, the linear momentum resp. the kinetic energy of a mass $\mathrm{d}M$ of fluid moving with velocity $\vec{\mathrm{v}}$ is $\mathrm{d}\vec{P} = (\mathrm{d}M)\vec{\mathrm{v}}$ resp. $\mathrm{d}K = \frac{1}{2}(\mathrm{d}M)\vec{\mathrm{v}}^2$, so that the associated specific density is $\mathrm{d}\vec{P}/\mathrm{d}M = \vec{\mathrm{v}}$ resp. $\mathrm{d}K/\mathrm{d}M = \frac{1}{2}\vec{\mathrm{v}}^2$.

Remark: These examples illustrate the fact that the tensorial nature—scalar, vector, tensor of higher rank—of the function associated with quantity $\mathcal{G}$ can be arbitrary.

For a material system Σ occupying at time t a volume $\mathcal{V}_\Sigma(t)$, Eq. (III.1) leads to

$$\mathcal{G}(t) = \int_{\mathcal{V}_\Sigma(t)} g_{\mathrm{m}}(t,\vec{r})\,\mathrm{d}M = \int_{\mathcal{V}_\Sigma(t)} g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\mathrm{d}^3\vec{r} \tag{III.2}$$

for the value of $\mathcal{G}$ of the system, where in the second identity $\rho(t,\vec{r}) = \mathrm{d}M/\mathrm{d}^3\vec{r}$ is the local mass density.

Let us now assume that the material system Σ is moving as part of a larger, flowing continuous medium. To find the substantial derivative $\mathrm{D}\mathcal{G}(t)/\mathrm{D}t$ of $\mathcal{G}(t)$, we shall first compute the variation $\delta\mathcal{G}$ for the material system Σ between times t and $t+\delta t$, where δt is assumed to be small. At the end of the calculation, we shall take the limit $\delta t \to 0$.

Going back to the regions (1), (2−), (2+) defined in Fig. III.1, one can write

$$\delta\mathcal{G} = \big(\mathcal{G}_1 + \mathcal{G}_{2+}\big)_{t+\delta t} - \big(\mathcal{G}_1 + \mathcal{G}_{2-}\big)_t = \delta\mathcal{G}_1 + \delta\mathcal{G}_2,$$

where the various indices denote the respective spatial domains and instants, and

$$\delta\mathcal{G}_1 \equiv \big(\mathcal{G}_1\big)_{t+\delta t} - \big(\mathcal{G}_1\big)_t, \qquad \delta\mathcal{G}_2 \equiv \big(\mathcal{G}_{2+}\big)_{t+\delta t} - \big(\mathcal{G}_{2-}\big)_t.$$

- $\delta\mathcal{G}_1$ represents the variation of $\mathcal{G}$ inside region (1) due to the non-stationarity of the fluid flow. Using a Taylor expansion to linear order in δt, one may write

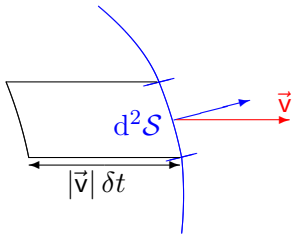
$$\delta\mathcal{G}_1 \simeq \frac{\mathrm{d}\mathcal{G}_1(t)}{\mathrm{d}t}\,\delta t = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\mathrm{d}^3\vec{r}\right]\delta t.$$

  Since the integration volume $\mathcal{V}$ is time-independent, one can pull the time derivative inside the integral, where it becomes a partial derivative, leading to

$$\delta\mathcal{G}_1 \simeq \int_{\mathcal{V}_\Sigma(t)} \frac{\partial}{\partial t}\big[g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\big]\,\mathrm{d}^3\vec{r}\;\delta t. \tag{III.3}$$

- $\delta\mathcal{G}_2$ represents the net amount of $\mathcal{G}$ traversing between t and $t+\delta t$ the control surface $\partial\mathcal{V}$, either leaving (region 2+) or entering (region 2−) the volume $\mathcal{V}$, where in the latter case the amount is counted negatively. This is precisely the *flux*—in the mathematical acceptation of the term—through the surface $\partial\mathcal{V}$, oriented towards the exterior, of an appropriate flux density for quantity $\mathcal{G}$.[(2)]

[(2)] This flux density can be read off Eq. (III.4), namely $g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})$.



Let $\vec{\mathrm{v}}$ denote the velocity of the continuous medium at position $\vec{r}$ at time t. The amount of quantity $\mathcal{G}$ that traverses in δt a surface element $\mathrm{d}^2\mathcal{S}$ situated in $\vec{r}$ equals the amount inside an elementary cylinder with base $\mathrm{d}^2\mathcal{S}$ and height $|\vec{\mathrm{v}}|\,\delta t$, i.e. $\mathrm{d}^3\mathcal{G} = g_{\mathrm{m}}\rho\,\mathrm{d}^3\mathcal{V}$, with $\mathrm{d}^3\mathcal{V} = |\mathrm{d}^2\vec{\mathcal{S}}\cdot\vec{\mathrm{v}}|\,\delta t$, where the vector $\mathrm{d}^2\vec{\mathcal{S}}$ is normal to the surface element.

Integrating over all surface elements all over the control surface, the net amount of quantity $\mathcal{G}$ flowing through $\partial\mathcal{V}$ thus reads

$$\delta\mathcal{G}_2 = \oint_{\partial\mathcal{V}} \mathrm{d}^3\mathcal{G} = \oint_{\partial\mathcal{V}} \left[g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot\mathrm{d}^2\vec{\mathcal{S}}\,\delta t. \tag{III.4}$$

All in all, Eqs. (III.3)–(III.4) yield after dividing by δt and taking the limit $\delta t \to 0$ the so-called *Reynolds transport theorem*:[(xxxi)]

$$\boxed{\frac{\mathrm{D}\mathcal{G}(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \frac{\partial}{\partial t}\left[g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\right]\mathrm{d}^3\vec{r} + \oint_{\partial\mathcal{V}} \left[g_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot\mathrm{d}^2\vec{\mathcal{S}}.} \tag{III.5}$$

The first term on the right hand side of this relation represents a *local* time derivative, similar to the first term in Eq. (I.18), for which reason it is often denoted $\partial\mathcal{G}/\partial t$. In contrast, the second term is of *convective* type, i.e. directly caused by the motion of matter, and represents the transport of $\mathcal{G}$.

Anticipating on the remainder of the Chapter, this theorem will help us as follows. The "usual" laws of dynamics are valid for closed, material systems Σ, rather than for open ones. Accordingly, these laws involve time derivatives "following the system in its motion", which is precisely what the material derivative $\mathrm{D}/\mathrm{D}t$ describes. Reynolds' transport theorem (III.5) expresses the latter, for extensive quantities $\mathcal{G}(t)$, in terms of local densities attached to fixed spatial positions, i.e. in Eulerian variables.

Remarks:

∗ When computing the flux through a surface element, we assumed that the latter is at rest, which defines the reference frame in which velocity $\vec{\mathrm{v}}(t,\vec{r})$ of the flowing medium is to be measured. Similarly, the flow velocity $\vec{\mathrm{v}}(t,\vec{r})$ in Reynolds' transport theorem is measured with respect to a reference frame in which the control surface $\partial\mathcal{V}$ is motionless.

∗ Since relation (III.5) is traditionally referred to as a theorem, one may wonder what are its assumptions. Obviously, the derivation of the result relies on the assumption that the specific density $g_{\mathrm{m}}(t,\vec{r})$ and the velocity field $\vec{\mathrm{v}}(t,\vec{r})$ are both continuous and differentiable, in agreement with the generic hypotheses in § I.2.2. Figure III.1 actually also embodies the hidden, but necessary assumption that the motion is continuous, which leads to the smooth evolution of the connected system of material points which are together inside the control surface $\partial\mathcal{V}$ at time t. Again, this follows from suitable properties of $\vec{\mathrm{v}}$.

∗ Accordingly, the Reynolds transport theorem (III.5) does not hold if the velocity field, or the specific density g_{m}, is discontinuous. As was already mentioned in § I.2.2, such discontinuities are however necessary to account for some phenomena (shock waves, boundary between two immiscible fluids...). In such cases, it will be necessary to reformulate the transport theorem to take into account the discontinuities.

(xxxi) *Reynolds'scher Transportsatz*

III.2 Mass and particle number conservation: continuity equation

The mass M and the particle number N of a closed non-relativistic system Σ remain constant in its motion:

$$\frac{\mathrm{D}M(t)}{\mathrm{D}t} = 0, \qquad \frac{\mathrm{D}N(t)}{\mathrm{D}t} = 0. \tag{III.6}$$

These conservation laws lead with the help of Reynolds' transport theorem to partial differential equations for some of the local fields that characterize a fluid flow.

III.2.1 Integral formulation

For an arbitrary control volume $\mathcal{V}$ delimited by surface $\partial\mathcal{V}$, the Reynolds transport theorem (III.5) with $\mathcal{G}(t) = M$, to which corresponds the specific density $g_{\mathrm{m}}(t,\vec{r}) = 1$, reads

$$\frac{\mathrm{D}M(t)}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} \rho(t,\vec{r})\,\mathrm{d}^3\vec{r}\right] + \oint_{\partial\mathcal{V}} \left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot \mathrm{d}^2\vec{\mathcal{S}} = 0. \tag{III.7}$$

That is, the time derivative of the mass contained in $\mathcal{V}$ equals the negative of the mass flow rate through $\partial\mathcal{V}$. In agreement with footnote (2), $\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})$ is the *mass flux density*,[(xxxii)] while its integral is the *mass flow rate*.[(xxxiii)]

Taking now $\mathcal{G}(t) = N$, the associated specific density is $g_{\mathrm{m}}(t,\vec{r}) = N/M$. Since the product of N/M with the mass density $\rho(t,\vec{r})$ is precisely the particle number density $n(t,\vec{r})$, Reynolds' theorem (III.5) leads to

$$\frac{\mathrm{D}N(t)}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} n(t,\vec{r})\,\mathrm{d}^3\vec{r}\right] + \oint_{\partial\mathcal{V}} \left[n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\cdot \mathrm{d}^2\vec{\mathcal{S}} = 0, \tag{III.8}$$

where $n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})$ represents the *particle number flux density*.[(xxxiv)]

Equation (III.7) resp. (III.8) constitutes the *integral formulation* of mass resp. particle number conservation.

Remarks:

∗ In the case of a steady motion, Eq. (III.7) shows that the net mass flow rate through an arbitrary closed geometrical surface $\partial\mathcal{V}$ vanishes. That is, the entrance of some amount of fluid into a (control) volume $\mathcal{V}$ must be compensated by the simultaneous departure of an equal mass from the volume.

∗ Considering "particle number" (and its conservation) in a fluid-based description is somewhat paradoxical. This number is to be understood as representative for any quantum number, carried at the microscopic level by particles, which is conserved in the fluid motion, like electric charge.

III.2.2 Local formulation

Since the control volume $\mathcal{V}$ in Eq. (III.7) resp. (III.8) is time-independent, the time derivative can be exchanged with the integration over volume. Besides, the surface integral can be transformed with the help of Gauss's theorem into a volume integral. All in all, this yields

$$\int_{\mathcal{V}} \left\{\frac{\partial\rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\right\}\mathrm{d}^3\vec{r} = 0,$$

resp.

$$\int_{\mathcal{V}} \left\{\frac{\partial n(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right]\right\}\mathrm{d}^3\vec{r} = 0.$$

(xxxii) *Massenstromdichte* (xxxiii) *Massenstrom* (xxxiv) *Teilchenstromdichte*

These identities hold for an arbitrary integration volume $\mathcal{V}$. Using the continuity of the respective integrands, one deduces the following so-called *continuity equations*:

$$\frac{\partial \rho(t,\vec{r})}{\partial t} + \vec{\nabla} \cdot \left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0 \tag{III.9}$$

resp.

$$\frac{\partial n(t,\vec{r})}{\partial t} + \vec{\nabla} \cdot \left[n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0. \tag{III.10}$$

Equation (III.9) represents the first of five dynamical (partial differential) equations which govern the evolution of a non-relativistic fluid flow.

Remarks:

$*$ The form of the continuity equation (III.9) does not depend on the properties of the flowing medium, as for instance whether dissipative effects play a significant role or not. This should be contrasted with the findings of the next two Sections.

$*$ In the case of a steady fluid flow, Eq. (III.9) gives $\vec{\nabla} \cdot \left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0$, i.e.

$$\rho(t,\vec{r})\,\vec{\nabla} \cdot \vec{\mathrm{v}}(t,\vec{r}) + \vec{\mathrm{v}}(t,\vec{r}) \cdot \vec{\nabla}\rho(t,\vec{r}) = 0.$$

Thus, the stationary flow of a *homogeneous* fluid, i.e. for which $\rho(t,\vec{r})$ is position independent, will be incompressible [$\vec{\nabla} \cdot \vec{\mathrm{v}}(t,\vec{r}) = 0$, cf. Eq. (II.15)].