Tutorial sheet 9

Discussion topic: Dynamical similarity and the Reynolds number.

18. Taylor–Couette flow. Measurement of shear viscosity

A Couette viscometer consists of an annular gap, filled with fluid, between two concentric cylinders with height L. The outer cylinder (radius R_2) rotates around the common axis with angular velocity Ω_2, while the inner cylinder (radius R_1) remains motionless. The motion of the fluid is assumed to be two-dimensional, laminar, incompressible, and steady.

Throughout this exercise, we use a system of cylinder coordinates (r, φ, z) with the physicists' usual convention, i.e. the corresponding basis vectors are are normalized to unity.

i. Check that the continuity equation leads to $\mathsf{v}^r = 0$, with v^r the radial component of the flow velocity.

ii. Prove that the Navier–Stokes equation lead to the equations

$$\frac{\mathsf{v}^\varphi(r)^2}{r} = \frac{1}{\rho}\frac{\partial \mathcal{P}(r)}{\partial r} \tag{1}$$

$$\frac{\partial^2 \mathsf{v}^\varphi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \mathsf{v}^\varphi(r)}{\partial r} - \frac{\mathsf{v}^\varphi(r)}{r^2} = 0. \tag{2}$$

What is the meaning of Eq. (1)? Solve Eq. (2) with the ansatz $\mathsf{v}^\varphi(r) = ar + \dfrac{b}{r}$.

iii. One can show (can you?) that the $r\varphi$-component of the stress tensor is given by

$$\sigma^{r\varphi} = \eta\left(\frac{1}{r}\frac{\partial \mathsf{v}^r}{\partial \varphi} + \frac{\partial \mathsf{v}^\varphi}{\partial r} - \frac{\mathsf{v}^\varphi}{r}\right).$$

Show that $\sigma^{r\varphi} = -\dfrac{2b\eta}{r^2}$, where b is the same coefficient as above.

iv. A torque $\mathcal{M}_z$ is measured at the surface of the inner cylinder. How can the shear viscosity η of the fluid be deduced from this measurement?
Numerical example: $R_1 = 10$ cm, $R_2 = 11$ cm, $L = 10$ cm, $\Omega_2 = 10$ rad·s^{-1} and $\mathcal{M}_z = 7{,}246\cdot 10^{-3}$ N·m.

19. Dimensional consideration for viscous flows in a tube

Consider the motion of a given fluid in a cylindrical tube of length L and of circular cross section under the action of a difference $\Delta\mathcal{P}$ between the pressures at the two ends of the tube. The relation between the pressure drop per unit length $\Delta\mathcal{P}/L$ and the magnitude of the mean velocity $\langle \mathsf{v}\rangle$—defined as the average over a cross section of the tube—is given by

$$\frac{\Delta\mathcal{P}}{L} = C\langle \mathsf{v}\rangle^n,$$

with C a constant that depends on the fluid mass density ρ, on the kinematic shear viscosity ν, and on the radius a of the tube cross section. n is a number which depends on the type of flow: $n = 1$ if the flow is laminar (this is the Hagen–Poiseuille law seen in the lecture), while measurements in turbulent flows by Hagen (1854) resp. Reynolds (1883) have given $n = 1.75$ resp. $n = 1.722$.

Assuming that C is—up to a pure number—a product of powers of ρ, ν and a, determine the exponents of these power laws using dimensional arguments.