Tutorial sheet 8

Discussion topic: What are the fundamental equations governing the dynamics of non-relativistic Newtonian fluids?

15. Heat diffusion

In a dissipative fluid at rest, the energy balance equation becomes

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla}\cdot\big[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\big]$$

with e the internal energy density, κ the heat capacity and T the temperature.

Assuming that $C \equiv \partial e/\partial T$ and κ are constant coefficients and introducing $\chi \equiv \kappa/C$, determine the temperature profile $T(t,\vec{r})$ for $z < 0$ with the boundary condition of a uniform, time-dependent temperature $T(t,z=0) = T_0\cos(\omega t)$ in the plane $z = 0$. At which depth is the amplitude of the temperature oscillations 10 % of that in the plane $z = 0$?

16. Flow of a Newtonian fluid down a constant slope

A layer of Newtonian fluid is flowing under the influence of gravity (acceleration g) down a slope inclined at an angle α from the horizontal. The fluid itself is assumed to have a constant thickness h, so that its free surface is a plane parallel to its bottom, and the flow is steady, laminar and incompressible. One further assumes that the pressure at the free surface of the fluid as well as "at the ends" at large $|x|$ is constant—i.e., the flow is entirely caused by gravity, not by a pressure gradient.
To fix notations, let x denote the direction along which the fluid flows, with the basis vector oriented downstream, and y be the direction perpendicular to x, oriented upwards.

i. Show that the flow velocity magnitude v and pressure $\mathcal{P}$ of the fluid obey the equations

$$\begin{cases} \dfrac{\partial \mathsf{v}}{\partial x} = 0 \\ \eta\triangle\mathsf{v} = -\rho g\sin\alpha \\ \dfrac{\partial\mathcal{P}}{\partial y} = -\rho g\cos\alpha, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} \mathsf{v} = 0 & \text{at } y = 0 \\ \dfrac{\partial \mathsf{v}}{\partial y} = 0 & \text{at } y = h \\ \mathcal{P} = \mathcal{P}_0 & \text{at } y = h. \end{cases} \tag{2}$$

Determine the pressure and then the velocity profile.

ii. Compute the rate of volume flow ("volumetric flux") across a surface $\mathcal{S}$ perpendicular to the x-direction.

17. Flow due to an oscillating plane boundary

Consider a rigid infinitely extended plane boundary ($y = 0$) that oscillates in its own plane with a sinusoidal velocity $U\cos(\omega t)\,\vec{e}_x$. The region $y > 0$ is filled with an incompressible Newtonian fluid with uniform kinematic shear viscosity ν. We shall assume that volume forces on the fluid are negligible, that the pressure is uniform and remains constant in time, and that the (laminar) fluid motion induced by the plane oscillations does not depend on the coordinates x, z.

i. Determine the flow velocity $\vec{\mathsf{v}}(t,y)$ and plot the resulting profile.

ii. What is the characteristic thickness of the fluid layer in the vicinity of the plane boundary that follows the oscillations? Comment on your result.