Tutorial sheet 7

13. Statics of rotating fluids

This exercise is strongly inspired by Chapter 13.3.3 of *Modern Classical Physics* by Roger D. Blandford and Kip S. Thorne.

Consider a fluid, bound by gravity, which is rotating rigidly, i.e. with a uniform angular velocity $\vec{\Omega}_0$ with respect to an inertial frame, around a given axis. In a reference frame that co-rotates with the fluid, the latter is at rest, and thus governed by the laws of hydrostatics—except that you now have to consider an additional term...

i. Relying on your knowledge from point mechanics, show that the usual equation of hydrostatics (in an inertial frame) is replaced in the co-rotating frame by

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\big[\Phi(\vec{r}) + \Phi_{\rm cen.}(\vec{r})\big], \tag{1}$$

where $\Phi_{\rm cen.}(\vec{r}) \equiv -\frac{1}{2}\big[\vec{\Omega}_0 \times \vec{r}\big]^2$ denotes the potential energy from which the centrifugal inertial force (density) derives, $\vec{f}_{\rm cen.} = -\rho\vec{\nabla}\Phi_{\rm cen.}$, while $\Phi(\vec{r})$ is the gravitational potential energy.

ii. Show that Eq. (1) implies that the equipotential lines of $\Phi + \Phi_{\rm cen.}$ coincide with the contours of constant mass density as well as with the isobars.

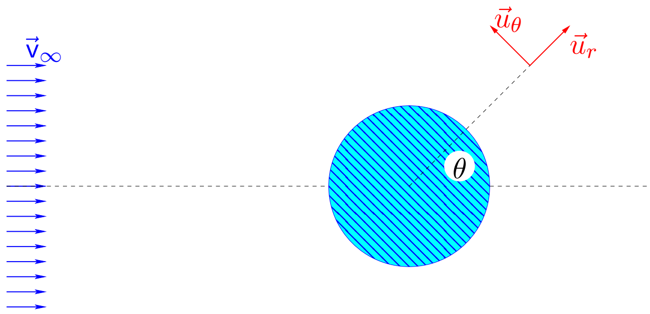
iii. Consider a slowly spinning fluid planet of mass M, assuming for the sake of simplicity that the mass is concentrated at the planet center, so that the gravitational potential is unaffected by the rotation. Let R_e resp. R_p denote the equatorial resp. polar radius of the planet, where $|R_e - R_p| \ll R_e \simeq R_p$, and g be the gravitational acceleration at the surface of the planet.
Using questions **i.** and **ii.**, show that the difference between the equatorial and polar radii is

$$R_e - R_p \simeq \frac{R_e^2|\vec{\Omega}_0|^2}{2g}.$$

Compute this difference in the case of Earth ($R_e \simeq 6.4 \times 10^3$ km)—which as everyone knows behaves as a fluid if you look at it long enough—and compare with the actual value.

14. Steady irrotational flow with a vortex. Magnus effect

The purpose of this exercise is to introduce a simplified model for the Magnus effect, which was discussed in the lectures.



One can show that the flow velocity of an incompressible perfect fluid around a cylinder of radius R at rest, with the uniform condition $\vec{v}(\vec{r}) = \vec{v}_\infty$ far from the cylinder—$\vec{v}_\infty$ being perpendicular to the cylinder axis—, is given by

$$\vec{\mathsf{v}}(r,\theta) = \mathsf{v}_\infty\left[\left(1-\frac{R^2}{r^2}\right)\cos\theta\,\vec{u}_r - \left(1+\frac{R^2}{r^2}\right)\sin\theta\,\vec{u}_\theta\right], \tag{2}$$

where (r,θ) are polar coordinates—the third dimension (z), along the cylinder axis, plays no role—with the origin at the center of the cylinder (see Figure) and $\vec{u}_r$, $\vec{u}_\theta$ unit length vectors.

One superposes to the velocity field (2) a vortex with circulation Γ, corresponding to a flow velocity

$$\vec{\mathsf{v}}(r,\theta) = \frac{\Gamma}{2\pi r}\vec{u}_\theta. \tag{3}$$

i. Let $C \equiv \Gamma/(4\pi R\mathsf{v}_\infty)$. Determine the points with vanishing velocity for the flow resulting from superposing (2) and (3).

Hint: Distinguish the two cases $C < 1$ and $C > 1$.

ii. How do the streamlines look like in each case? Comment on the physical meaning of the result.

iii. Express the force per unit length $\mathrm{d}\vec{F}/\mathrm{d}z$ exerted on the cylinder by the flow (2)+(3) as function of Γ, v_∞ and the mass density ρ of the fluid.