Tutorial sheet 6

Discussion topic: What is Kelvin's circulation theorem? What does it imply for the vorticity?

11. Differential form of Thomson's theorem

i. Consider the motion of a perfect barotropic fluid with conservative external volume forces. Show that the vorticity vector field $\vec{\omega}(t,\vec{r})$ obeys the equation

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} = \vec{\nabla}\times\big[\vec{\mathsf{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})\big].$$

ii. Stationary vortex: Let $\vec{\omega}(t,\vec{r}) = A\,\delta(x^1)\,\delta(x^2)\,\vec{\mathrm{e}}_3$ be the vorticity field in a fluid, with A a real constant and $\{x^i\}$ Cartesian coordinates. Determine the corresponding flow velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

Hint: You should invoke symmetry arguments and Stokes' theorem. A useful formal analogy is provided by the Maxwell–Ampère equation of magnetostatics.

12. Model of a tornado

In a simplified approach, one may model a tornado as the steady incompressible flow of a perfect fluid—air—with mass density $\rho = 1.3\ \mathrm{kg\cdot m^{-3}}$, with a vorticity $\vec{\omega}(\vec{r}) = \omega(\vec{r})\,\vec{\mathrm{e}}_3$ which remains uniform inside a cylinder—the "eye" of the tornado—with (vertical) axis along $\vec{\mathrm{e}}_3$ and a finite radius $a = 50$ m, and vanishes outside.

i. Express the velocity $\mathsf{v}(r) \equiv \big|\vec{\mathsf{v}}(\vec{r})\big|$ at a distance $r = |\vec{r}|$ from the axis as a function of r and and the velocity $\mathsf{v}_a \equiv \mathsf{v}(r\!=\!a)$ at the edge of the eye.
Compute ω inside the eye, assuming $\mathsf{v}_a = 180$ km/h.

ii. Show that for $r > a$ the tornado is equivalent to a vortex at $x^1 = x^2 = 0$ (as in exercise **11.ii**). What is the circulation around a closed curve circling this equivalent vortex?

iii. Assuming that the pressure $\mathcal{P}$ far from the tornado equals the "normal" atmospheric pressure $\mathcal{P}_0$, determine $\mathcal{P}(r)$ for $r > a$. Compute the barometric depression $\Delta\mathcal{P} \equiv \mathcal{P}_0 - \mathcal{P}$ at the edge of the eye.
Consider a horizontal roof made of a material with mass surface density 100 kg/m^2: is it endangered by the tornado?