Tutorial sheet 4

Discussion topic: What are the basic equations governing the dynamics of perfect fluids?

6. Simplified model of star

In an oversimplified approach, one may model a star as a sphere of fluid—a plasma—with uniform mass density ρ. This fluid is in mechanical equilibrium under the influence of pressure $\mathcal{P}$ and gravity. Throughout this exercise, the rotation of the star is neglected.

For this exercise you need almost no knowledge from the Hydrodynamics lecture, only some understanding of pressure forces in a static fluid.

i. Determine the gravitational field at a distance r from the center of the star.

ii. Assuming that the pressure only depend on r, write down the equation expressing the mechanical equilibrium of the fluid. Determine the resulting function $\mathcal{P}(r)$. Compute the pressure at the star center as function of the mass M and radius R of the star. Calculate the numerical value of this pressure for $M = 2 \times 10^{30}$ kg (solar mass) and $R = 7 \times 10^{8}$ m (solar radius).

iii. The matter constituting the star is assumed to be an electrically neutral mixture of hydrogen nuclei and electrons. Show that the order of magnitude of the total particle number density of that plasma is $n \approx 2\rho/m_p$, with m_p the proton mass. Estimate the temperature at the center of the sun.
Hint: $m_p = 1.6 \times 10^{-27}$ kg; $k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J \cdot K^{-1}}$.

7. Symmetry of the stress tensor

Let $\boldsymbol{\sigma}_{ij} = -\mathbf{T}_{ij}$ denote the Cartesian components of the stress tensor in a continuous medium. Consider an infinitesimal cube of medium, whose edges (length $\mathrm{d}\ell$) are parallel to the axes of the coordinate system.

i. Explain why the k-component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the continuous medium obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\mathbf{T}_{ij}(\mathrm{d}\ell)^3$, with ϵ_{ijk} the usual Levi-Civita symbol.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the continuum mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?