Tutorial sheet 3

Discussion topic: What is the Reynolds transport theorem (and its utility)?

4. Example of a motion

Consider the motion defined in a system of Cartesian coordinates with basis vectors $(\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_3)$ by the velocity field with components

$$\mathsf{v}^1(t,\vec{r}) = f_1(t,x^2), \quad \mathsf{v}^2(t,\vec{r}) = f_2(t,x^1), \quad \mathsf{v}^3(t,\vec{r}) = 0,$$

with f_1, f_2 two continuously differentiable functions.

Compute the strain rate tensor $\mathbf{D}(t,\vec{r})$ for this motion. What is the volume expansion rate? Give the rotation rate tensor $\mathbf{R}(t,\vec{r})$ and the vorticity vector. Under which condition(s) on the functions f_1, f_2 does the motion become irrotational?

5. Two motions with cylindrical symmetry

In this exercise, we use a system of cylindrical coordinates (r,θ,z).

i. Pointlike source

Consider the fluid motion defined for $r \neq 0$ by the velocity field

$$\mathsf{v}^r(t,\vec{r}) = \frac{f(t)}{r}, \quad \mathsf{v}^\theta(t,\vec{r}) = 0, \quad \mathsf{v}^z(t,\vec{r}) = 0,$$

with f some scalar function.

a) Compute the volume expansion rate and the vorticity vector.

b) Mathematically, the velocity field is singular at $r = 0$. Thinking of the velocity profile, what do you have *physically* at that point if $f(t) > 0$? if $f(t) < 0$?

ii. Pointlike vortex

Consider now the fluid motion defined for $r \neq 0$ by the velocity field

$$\vec{\mathsf{v}}(t,\vec{r}) = \frac{\Gamma}{2\pi r}\vec{u}_\theta, \quad \Gamma \in \mathbb{R},$$

where $\vec{u}_\theta$ denotes a unit vector in the orthoradial direction.[1] Give the corresponding volume expansion rate and vorticity vector. Compute the *circulation* of the velocity field along a closed curve circling the z-axis. For which physical phenomenon could this motion be a (very crude!) model?

iii. The velocity fields of questions **i.** — assuming that $f(t)$ is time-independent — and **ii.** are analogous to the electrical or magnetic fields created by simple (stationary) distributions of electric charges or currents. Do you see which?

[1] That is, $\vec{u}_\theta$ is in the plane perpendicular to the z-axis and orthogonal to the radial direction away from the z-axis.