Tutorial sheet 12

Discussion topic: Turbulence in fluids: what is it? why does it require a Reynolds number larger than some critical value to develop? In fully developed turbulence, what are the mean flow, the fluctuating flow, the Reynolds stress tensor, the energy cascade?

22. Inviscid Burgers equation

The purpose of this exercise is to show how an innocent-looking—yet non-linear—partial differential equation with a smooth initial condition may lead after finite amount of time to a discontinuity, i.e. a shock wave.

Neglecting the pressure term in the one-dimensional Euler equation leads to the so-called *inviscid Burgers*[1] *equation*

$$\frac{\partial \mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x} = 0.$$

i. Show that the solution with (arbitrary) given initial condition $\mathsf{v}(0,x)$ for $x \in \mathbb{R}$ obeys the implicit equation $\mathsf{v}(0,x) = \mathsf{v}\big(t, x + \mathsf{v}(0,x)\,t\big)$.

Hint: `http://en.wikipedia.org/wiki/Burgers'_equation`

ii. Consider the initial condition $\mathsf{v}(0,x) = \mathsf{v}_0\,\mathrm{e}^{-(x/x_0)^2}$ with v_0 and x_0 two real numbers. Show that the flow velocity becomes discontinuous at time $t = \sqrt{\mathrm{e}/2}\,x_0/\mathsf{v}_0$, namely at $x = x_0\sqrt{2}$.

23. Dynamics of the mean flow in fully developed turbulence

The velocity field resp. pressure for an incompressible turbulent flow is split into an average and a fluctuating part as

$$\vec{\mathsf{v}}(t,\vec{r}) = \overline{\vec{\mathsf{v}}}(t,\vec{r}) + \vec{\mathsf{v}}'(t,\vec{r}) \quad \text{resp.} \quad \mathcal{P}(t,\vec{r}) = \overline{\mathcal{P}}(t,\vec{r}) + \mathcal{P}'(t,\vec{r}),$$

where the motion with $\overline{\vec{\mathsf{v}}}$, $\overline{\mathcal{P}}$ is referred to as "mean flow". For the sake of simplicity, a system of Cartesian coordinates is being assumed—the components of the gradient thus involve partial derivatives, instead of the more general covariant derivatives. Throughout the exercise, Einstein's summation convention over repeated indices is used.

Check that the incompressible Navier–Stokes equation obeyed by $\vec{\mathsf{v}}$ and $\mathcal{P}$ leads for the mean-flow quantities to the equation

$$\frac{\partial \overline{\mathsf{v}^i}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{\mathsf{v}^i} = -\frac{1}{\rho}\frac{\partial \overline{\mathcal{P}}}{\partial x_i} - \frac{\partial \overline{\mathsf{v}'^i\mathsf{v}'^j}}{\partial x^j} + \nu\triangle\overline{\mathsf{v}^i}. \tag{1}$$

Show that this gives for the kinetic energy per unit mass $\overline{k} \equiv \frac{1}{2}\overline{\vec{\mathsf{v}}}^2$ associated with the mean flow the evolution equation

$$\frac{\partial \overline{k}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{k} = -\frac{\partial}{\partial x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\overline{\mathsf{v}^j} + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathsf{v}_i}\right] + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathbf{S}_{ij}} \tag{2}$$

with $\overline{\mathbf{S}^{ij}} \equiv \dfrac{1}{2}\left(\dfrac{\partial\overline{\mathsf{v}^i}}{\partial x_j} + \dfrac{\partial\overline{\mathsf{v}^j}}{\partial x_i} - \dfrac{2}{3}g^{ij}\vec{\nabla}\cdot\overline{\vec{\mathsf{v}}}\right)$ the components of the (mean) rate-of-shear tensor.

[1] J. Burgers, 1895–1981