Fundamental equations of nonrelativistic fluid dynamics

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Fundamental equations of nonrelativistic fluid dynamics

- Reynolds' transport theorem
- Equations of motion of a perfect fluid

May 11 & 18

- Equations of motion of a Newtonian fluid

June 8 & 15

Throughout these slides, Cartesian coordinates ($\{x_j\}$, $\{\mathrm{v}_j\}$, ...) are used:

$$x_j = x^j \ , \ \mathrm{v}_j = \mathrm{v}^j \ldots \text{ for } j = 1,2,3$$

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Momentum law for a real fluid

In a real ($\neq$ perfect) fluid, friction forces and heat transport are no longer neglected.

The momentum-flux tensor can take a more complicate form than in the perfect-fluid case (cf. slide 13):

$$\mathsf{T}^{ij}(t,\vec{r}) \equiv \mathcal{P}(t,\vec{r})\,\delta^{ij} + \rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r}) + \pi^{ij}(t,\vec{r})$$

where the "viscous stress tensor" $\pi^{ij}(t,\vec{r})$ is function of the derivatives of the velocity field (and possibly of other thermodynamical quantities).

For a "Newtonian fluid":

$$\pi^{ij}(t,\vec{r}) \equiv -\,\eta(t,\vec{r})\Bigg[\overbrace{\frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial\mathsf{v}^j(t,\vec{r})}{\partial x^i} - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\delta^{ij}(t,\vec{r})}^{\text{traceless}}\Bigg]$$
$$-\,\underbrace{\zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\delta^{ij}(t,\vec{r})}_{\text{diagonal}}$$

Momentum law for a Newtonian fluid

... reads
$$\frac{\partial}{\partial t}\big[\rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\big] + \sum_{j=1}^{3}\frac{\partial \mathbf{T}^{ij}(t,\vec{r})}{\partial x^j} = f_V^i(t,\vec{r})$$

where

$$\mathbf{T}^{ij}(t,\vec{r}) \equiv \mathcal{P}(t,\vec{r})\,\delta^{ij} + \rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r}) + \pi^{ij}(t,\vec{r})$$

with

$$\pi^{ij}(t,\vec{r}) \equiv -\,\eta(t,\vec{r})\left[\frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}^j(t,\vec{r})}{\partial x^i} - \frac{2}{3}\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\delta^{ij}(t,\vec{r})\right] - \zeta(t,\vec{r})\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\delta^{ij}(t,\vec{r})$$

- η: shear viscosity — multiplies the shear stress tensor

- ζ: bulk (or second) viscosity — multiplies the volume expansion rate $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})$: only relevant in compressible flows!

Momentum law for a Newtonian fluid

Navier–Stokes equation

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t}+\left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right\}=-\vec{\nabla}\mathcal{P}(t,\vec{r})+\vec{f}_V(t,\vec{r})+\eta\triangle\vec{\mathrm{v}}(t,\vec{r})$$
$$+\left(\zeta+\frac{\eta}{3}\right)\vec{\nabla}\left[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\right]$$

- η: shear viscosity
- ζ: bulk viscosity
- $\nu\equiv\eta/\rho$: kinematic shear viscosity

η, ζ: Pa·s (= "Poiseuille" [pronounce: ≈ Puasöj]); ν: $m^2 \cdot s^{-1}$

Momentum law for a Newtonian fluid

Navier–Stokes equation

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t}+\left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right\}=-\vec{\nabla}\mathcal{P}(t,\vec{r})+\vec{f}_V(t,\vec{r})+\eta\triangle\vec{\mathrm{v}}(t,\vec{r})$$
$$+\left(\zeta+\frac{\eta}{3}\right)\vec{\nabla}\left[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\right]$$

Boundary conditions:

- at a wall or an obstacle, "impermeability" + "no slip" (friction!)

☞ relative velocity vanishes

Force exerted by a Newtonian fluid

Stress vector (cf. lecture I) on a small surface element

$$\vec{T}_s(t,\vec{r}) = \sum_{i,j=1}^{3} \left\{ \left[-\mathcal{P}(t,\vec{r}) + \left(\zeta - \frac{2}{3}\eta\right) \vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}) \right] \delta^i{}_j \right.$$
$$\left. + \eta \left[\frac{\partial \mathrm{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathrm{v}^j(t,\vec{r})}{\partial x^i} \right] \right\} n^j(\vec{r})\, \vec{\mathrm{e}}_i$$

with $\vec{\mathrm{e}}_n(\vec{r}) \equiv \sum_i n^i(\vec{r})\, \vec{\mathrm{e}}_i$ normal unit vector to the surface element.

- First line: normal stress: hydrostatic pressure + viscous part (absent in an incompressible flow).
- Second line: tangential stress

Energy balance in a Newtonian fluid

... in the presence of conservative forces $\vec{f}_V(\vec{r}) = -\rho(\vec{r})\vec{\nabla}\Phi(\vec{r})$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right] + \vec{\nabla}\cdot\left\{\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r}) - \eta\left[\left(\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right)\vec{\mathrm{v}}(t,\vec{r}) + \vec{\nabla}\left(\frac{\vec{\mathrm{v}}(t,\vec{r})^2}{2}\right)\right] - \left(\zeta - \frac{2\eta}{3}\right)\left[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r}) - \kappa\vec{\nabla}T(t,\vec{r})\right\} = 0$$

- Viscous forces (shear + bulk) exert work
- **Heat is transported: heat current** $\vec{j}_Q(t,\vec{r}) = -\kappa\vec{\nabla}T(t,\vec{r})$

Fourier's law

Entropy balance in a Newtonian fluid

$$\frac{\partial s(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[s(t,\vec{r})\,\vec{\mathsf{v}}(t,\vec{r}) - \kappa\frac{\vec{\nabla}T(t,\vec{r})}{T(t,\vec{r})}\right] = \frac{1}{T(t,\vec{r})}\left\{2\eta\,\mathbf{S}(t,\vec{r}):\mathbf{S}(t,\vec{r}) + \zeta\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]^2 + \kappa\frac{\big[\vec{\nabla}T(t,\vec{r})\big]^2}{T(t,\vec{r})}\right\}$$

with
$$\mathbf{S}(t,\vec{r}):\mathbf{S}(t,\vec{r}) \equiv \sum_{i,j=1}^{3}\mathbf{S}_{ij}(t,\vec{r})\mathbf{S}_{ij}(t,\vec{r})$$

$$\mathbf{S}_{ij}(t,\vec{r}) \equiv \frac{1}{2}\left[\frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}^j(t,\vec{r})}{\partial x^i} - \frac{2\delta^{ij}}{3}\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\right]$$

Entropy increases (second law!) if η, ζ, and κ are positive.