Fundamental equations of nonrelativistic fluid dynamics

Fundamental equations of nonrelativistic fluid dynamics

- Reynolds' transport theorem

- Equations of motion of a perfect fluid

May 11 & 18

- Equations of motion of a Newtonian fluid

June 8 & 15

Throughout these slides, Cartesian coordinates ($\{x_j\}$, $\{\mathrm{v}_j\}$, ...) are used:

$$x_j = x^j\ ,\ \mathrm{v}_j = \mathrm{v}^j \ldots \text{ for } j = 1,2,3$$

Reynolds' transport theorem

A simple issue:

- In the Eulerian description of a fluid motion, the matter contained inside a geometrical volume ("control volume") is constantly changing

  ☞ open system

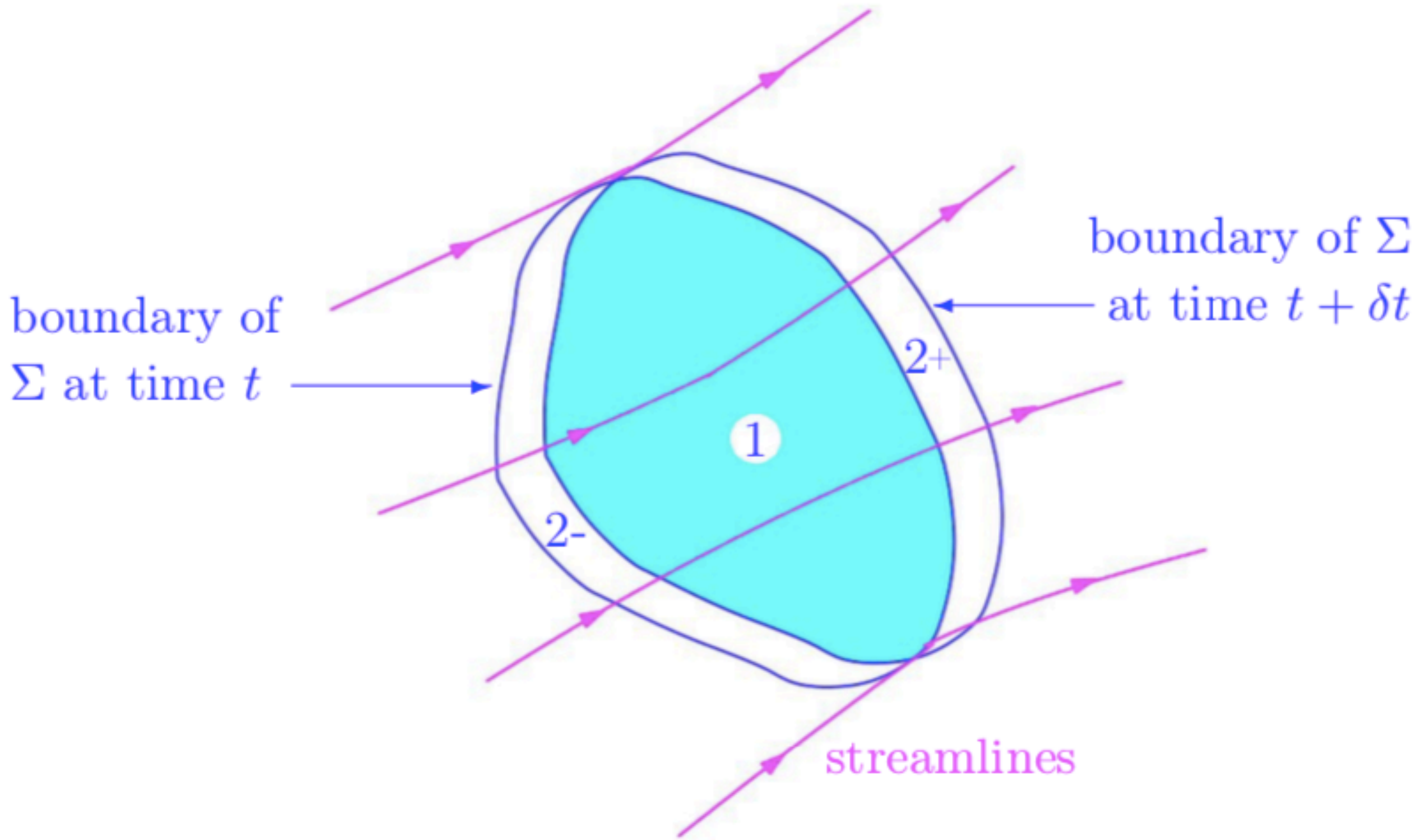
- The classical equations describing the motion of mechanical systems are usually formulated for closed systems that are followed in their evolution

  ☞ Newton's 2nd law, conservation of energy & of particle number...

- How do we derive equations implementing the key physics principles in terms of local "Eulerian" fields?

Reynolds' transport theorem

Idea: follow the system that occupies a control volume $\mathcal{V}$ at time t in its motion between t and $t+\delta t$:



Reynolds' transport theorem

Idea: follow the system that occupies a control volume $\mathcal{V}$ at time t in its motion between t and $t+\delta t$.

☞ rate of change of an extensive (physical!) quantity $\mathcal{G}$:

$$\frac{\mathrm{D}\mathcal{G}(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \frac{\partial}{\partial t}\big[\mathscr{g}_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\big]\,\mathrm{d}^3\vec{r} + \oint_{\partial\mathcal{V}} \big[\mathscr{g}_{\mathrm{m}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\big]\cdot\mathrm{d}^2\vec{\mathcal{S}}$$

where $\mathscr{g}_{\mathrm{m}}(t,\vec{r})$ is the local amount of $\mathcal{G}$ per unit mass.

- 1st term: local rate of change (due to non-stationarity) inside $\mathcal{V}$;
- 2nd term: flux of $\mathcal{G}$ through the surface $\partial\mathcal{V}$: convective term.

Conservation of mass

- Global formulation: $$\frac{\mathrm{D}M(t)}{\mathrm{D}t} = 0$$

- Local formulation:

Reynolds' transport theorem gives: (specific density of mass = 1!)

$$\frac{\mathrm{D}M(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \frac{\partial\rho(t,\vec{r})}{\partial t}\,\mathrm{d}^3\vec{r} + \oint_{\partial\mathcal{V}} \big[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\big]\cdot\mathrm{d}^2\vec{\mathcal{S}}$$

Gauss theorem →

$$= \int_{\mathcal{V}} \left[\frac{\partial\rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\big[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\big]\right]\mathrm{d}^3\vec{r}$$

Vanishes for every (control) volume $\mathcal{V}$ ☞ "continuity equation"

$$\frac{\partial\rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\big[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\big] = 0$$

Newton's second law (momentum law)

- Global formulation:

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \vec{F}(t)$$

- Local formulation:

Reynolds' transport theorem: specific density of $\vec{P}$: $\vec{\mathrm{v}}(\vec{r})$

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} \vec{\mathrm{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\mathrm{d}^3\vec{r}\right] + \oint_{\partial\mathcal{V}} \vec{\mathrm{v}}(t,\vec{r})\,\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\cdot\mathrm{d}^2\vec{\mathcal{S}}$$

$$= \int_{\mathcal{V}} \rho(t,\vec{r}) \underbrace{\left[\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right]}_{\frac{\mathrm{D}\vec{\mathrm{v}}(t,\vec{r})}{\mathrm{D}t}} \mathrm{d}^3\vec{r}$$

Using Gauss' theorem
& the continuity equation

Newton's second law (momentum law)

- Global formulation: $$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \vec{F}(t)$$

- Local formulation:

Reynolds' transport theorem: specific density of $\vec{P}$: $\vec{\mathsf{v}}(\vec{r})$

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \rho(t,\vec{r}) \left[\frac{\partial \vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla}\right] \vec{\mathsf{v}}(t,\vec{r}) \right] \mathrm{d}^3\vec{r}$$

Forces:

$$\vec{F}(t) = \int_{\mathcal{V}} \underbrace{\vec{f}_V(t,\vec{r})}\, \mathrm{d}^3\vec{r} + \oint_{\partial\mathcal{V}} \underbrace{\vec{\mathcal{T}}_s(t,\vec{r})}\, \mathrm{d}^2\mathcal{S}$$

Volume forces:
external (gravity...)

Surface forces:
inner to the fluid
☞ model of the fluid

A fluid model: the perfect fluid

A **perfect fluid** is a fluid in which there are no shear stresses nor heat conduction.

Does this exist? We'll discuss that again when studying non-perfect fluids, to see which properties (= "transport coefficients") have to be set to zero...

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A fluid model: the perfect fluid

A **perfect fluid** is a fluid in which there are no shear stresses nor heat conduction.

No shear stress: $\vec{T}_s(t,\vec{r}) = -\underbrace{\mathcal{P}(t,\vec{r})}_{\text{pressure}}\,\vec{\rm e}_n(\vec{r})$

leading to

$$\oint_{\partial\mathcal{V}} \vec{T}_s(t,\vec{r})\,{\rm d}^2\mathcal{S} = -\oint_{\partial\mathcal{V}} \mathcal{P}(t,\vec{r})\,\vec{\rm e}_n(\vec{r})\,{\rm d}^2\mathcal{S} \underset{\text{divergence theorem}}{=} -\int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,{\rm d}^3\vec{r}$$

Momentum law for a perfect fluid

- Global formulation: $$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \vec{F}(t)$$

- Local formulation:

Reynolds' transport theorem: specific density of $\vec{P}$: $\vec{\mathsf{v}}(\vec{r})$

$$\frac{\mathrm{D}\vec{P}(t)}{\mathrm{D}t} = \int_{\mathcal{V}} \rho(t,\vec{r}) \left[\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r}) \right] \mathrm{d}^3\vec{r}$$

=

Forces: $$\vec{F}(t) = \int_{\mathcal{V}} \vec{f}_V(t,\vec{r})\,\mathrm{d}^3\vec{r} - \int_{\mathcal{V}} \vec{\nabla}\mathcal{P}(t,\vec{r})\,\mathrm{d}^3\vec{r}$$

$$\rho(t,\vec{r}) \left[\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r}) \right] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_V(t,\vec{r})$$

Euler equation

Euler equation

$$\rho(t,\vec{r})\left[\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t}+\left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right]=-\vec{\nabla}\mathcal{P}(t,\vec{r})+\vec{f}_V(t,\vec{r})$$

Partial differential equation ("PDE"):

need to specify boundary conditions

- Impermeability at a wall / an obstacle: no normal component for the relative velocity

  but no condition on the tangential component

- At infinity: "user-specified" (for instance: uniform)

Euler equation

$$\rho(t,\vec{r})\left[\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t}+\left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right]=-\vec{\nabla}\mathcal{P}(t,\vec{r})+\vec{f}_V(t,\vec{r})$$

Alternative formulations:

- with the density of volume forces per unit mass $\vec{a}_V\equiv\vec{f}_V/\rho$
  - with the material derivative:

$$\frac{\mathrm{D}\vec{\mathrm{v}}(t,\vec{r})}{\mathrm{D}t}=-\frac{\vec{\nabla}\mathcal{P}(t,\vec{r})}{\rho(t,\vec{r})}+\vec{a}_V(t,\vec{r})$$

simple interpretation

  - with the vorticity:

$$\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t}+\vec{\nabla}\left[\frac{\vec{\mathrm{v}}(t,\vec{r})^2}{2}\right]-\vec{\mathrm{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})=-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r})+\vec{a}_V(t,\vec{r})$$

will be exploited in upcoming lectures

Euler equation

$$\rho(t,\vec{r})\left[\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t}+\left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r})\right]=-\vec{\nabla}\mathcal{P}(t,\vec{r})+\vec{f}_V(t,\vec{r})$$

Alternative formulations:

- as a balance equation:

$$\frac{\partial}{\partial t}\left[\rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\right]+\sum_{j=1}^{3}\frac{\partial\mathbf{T}^{ij}(t,\vec{r})}{\partial x^j}=f_V^i(t,\vec{r})$$

with the momentum flux tensor (= how much p^i flows in direction j)

$$\mathbf{T}^{ij}(t,\vec{r})\equiv\mathcal{P}(t,\vec{r})\,\delta^{ij}+\rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})$$

and the momentum density $\rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})$.

time derivative of density + divergence of current = source term

Energy conservation in perfect fluids

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right]$$
$$+\ \vec{\nabla}\cdot\left\{\left[\frac{1}{2}\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})^2 + e(t,\vec{r}) + \mathcal{P}(t,\vec{r}) + \rho(t,\vec{r})\,\Phi(t,\vec{r})\right]\vec{\mathrm{v}}(t,\vec{r})\right\} = 0$$

with $e(t,\vec{r})$: internal energy density

and $\Phi(t,\vec{r})$: external potential, such that $\vec{f}_V(t,\vec{r}) = -\rho(t,\vec{r})\,\vec{\nabla}\Phi(t,\vec{r})$

Work of pressure forces

Entropy conservation in perfect fluids

A by now familiar form:

$$\frac{\partial s(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[s(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0$$

with $s(t,\vec{r})$: entropy density