Tutorial sheet 9

17. Dimensionless equations of motion for sea surface waves

This exercise is partly a continuation of the lecture (June 24) on linear sea surface waves, which you should check if you are not sure of the notations employed.

The equations of motion governing gravity waves at the free surface of an incompressible perfect liquid (ocean/sea water) in a gravity field $-g\,\vec{e}_z$ are

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0, \tag{1a}$$

$$\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \left[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathsf{v}}(t,\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(t,\vec{r}) - g\,\vec{e}_z, \tag{1b}$$

with the boundary conditions $\mathsf{v}_z(t,x,z\!=\!0) = 0$ at the sea bottom;

$$\mathsf{v}_z\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \frac{\partial\delta h(t,x)}{\partial t} + \mathsf{v}_x(t,\vec{r})\frac{\partial\delta h(t,x)}{\partial x} \tag{1c}$$

at the free surface, situated at $z = h_0 + \delta h(t,x)$; and a uniform pressure at that same free surface, which may be re-expressed as

$$\mathcal{P}\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \rho g\,\delta h(t,x) + \mathcal{P}_0 \tag{1d}$$

with $\mathcal{P}_0$ a constant whose precise value is irrelevant. As in the lecture, the problem is assumed to be two-dimensional.

i. We introduce characteristic scales for various quantities: δh_c for the amplitude of the surface deformation; L_c for lengths along the horizontal direction x; and t_c for durations—in practice, the "good" choice would be $t_c = L_c/c_s$ with c_s the speed of sound, yet this is irrelevant here. With their help, we define dimensionless variables

$$t^* \equiv \frac{t}{t_c}, \quad x^* \equiv \frac{x}{L_c}, \quad z^* \equiv \frac{z}{L_c},$$

and fields:

$$\delta h^* \equiv \frac{\delta h}{\delta h_c}, \quad \mathsf{v}_x^* \equiv \frac{\mathsf{v}_x}{\delta h_c/t_c}, \quad \mathsf{v}_z^* \equiv \frac{\mathsf{v}_z}{\delta h_c/t_c}, \quad \mathcal{P}^* \equiv \frac{\mathcal{P}-\mathcal{P}_0}{\rho\,\delta h_c L_c/t_c^2}.$$

Considering the latter as functions of the reduced variables t^*, x^*, z^*, rewrite the equations (1a)–(1d), making use of the dimensionless numbers

$$\mathrm{Fr} \equiv \frac{\sqrt{L_c/g}}{t_c}, \quad \varepsilon \equiv \frac{\delta h_c}{L_c}, \quad \delta \equiv \frac{h_0}{L_c}.$$

What does the parameter ε control (mathematically)? and the parameter δ (physically)?

ii. Let us from now on assume that the flow is irrotational.

a) Let $\varphi(t^*,x^*,z^*)$ denote the velocity potential, such that $\vec{\mathsf{v}}^* = -\vec{\nabla}^*\varphi$. Assuming that φ can be written as an infinite series in z^*

$$\varphi(t^*,x^*,z^*) = \sum_{n=0}^{\infty} z^{*n}\varphi_n(t^*,x^*), \tag{2}$$

show that the functions φ_n obey a set of coupled equations, such that every φ_n with even resp. odd n can ultimately be related to φ_0 resp. φ_1.

Hint: Remember that the velocity potential obeys a celebrated differential equation.

b) Inspecting the series (2), you can equate φ_1 to one of the components of the (dimensionless) velocity $\vec{\mathsf{v}}^*$ at $z^* = 0$. Using the boundary conditions for the flow, deduce that all φ_n with odd n vanish, and that the velocity potential is eventually given by

$$\varphi(t^*, x^*, z^*) = \varphi_0(t^*, x^*) - \frac{z^{*2}}{2} \frac{\partial^2 \varphi_0(t^*, x^*)}{\partial x^{*2}} + \frac{z^{*4}}{4!} \frac{\partial^4 \varphi_0(t^*, x^*)}{\partial x^{*4}} + \ldots$$

Introducing the notation $\mathsf{u}(t^*, x^*) \equiv -\partial\varphi_0(t^*, x^*)/\partial x^*$, express $\mathsf{v}_x^*(t^*, x^*, z^*)$ and $\mathsf{v}_z^*(t^*, x^*, z^*)$ as a function of $\mathsf{u}(t^*, x^*)$ up to order z^{*3}.

iii. a) Show that you can combine some of the dimensionless equations found in question **i.** into

$$\frac{\partial \mathsf{v}_x^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^* \frac{\partial \mathsf{v}_x^*}{\partial x^*} + \mathsf{v}_z^* \frac{\partial \mathsf{v}_z^*}{\partial x^*}\right) + \frac{1}{\mathrm{Fr}^2} \frac{\partial \delta h^*}{\partial x^*} = 0. \tag{3}$$

b) Neglecting the term of order ε in the previous equation and truncating the expressions of v_x^*, v_z^* of question **ii.b)** to linear order in z^*, show that a proper combination of Eq. (3) and the boundary conditions leads to a simple partial differential equation for u. What do you recognize?