# Tutorial sheet 8

## 15. Two-dimensional potential flow. Teapot effect

Consider a steady two-dimensional potential flow with velocity $\vec{\mathsf{v}}(x,y)$, with $(x,y)$ Cartesian coordinates. The associated complex velocity potential is denoted $\phi(z)$, where $z = x + \mathrm{i}y$.

**i.** Consider the complex potential $\phi(z) = Az^n$ with $A \in \mathbb{R}$ and $n \geq 1/2$. Show that this potential allows you to describe the flow velocity in the sector $\widehat{\mathcal{E}}$ delimited by two walls making an angle $\alpha = \pi/n$.

**ii.** What can you say about the flow velocity in the vicinity of the end-corner of the sector $\widehat{\mathcal{E}}$?

*Hint*: Distinguish the cases $\alpha < \pi$ and $\alpha > \pi$.

**iii. Teapot effect**

If one tries to pour tea "carefully" from a teapot, one will observe that the liquid will trickle along the lower side of the nozzle, instead of falling down into the cup waiting below. Explain this phenomenon using the flow profile introduced above (in the case $\alpha > \pi$) and the Bernoulli equation.

*Literature*: Jearl Walker, Scientific American, Oct. 1984 (= Spektrum der Wissenschaft, Feb. 1985).

**iv.** Assuming now that you are using the potential $\phi(z) = Az^n$ to model the flow of a river, which qualitative behavior can you anticipate for its bank?

## 16. Conformal deformations of a two-dimensional potential flow

**i.** Consider the mapping ("Joukowsky transform")

$$\mathsf{f}\colon \mathbb{C} \to \mathbb{C}, \qquad z \mapsto Z = \mathsf{f}(z) \equiv z + \frac{R_{\mathrm{J}}^2}{z} \tag{1}$$

where $R_{\mathrm{J}} \in \mathbb{R}_+^*$. This mapping is holomorphic for $z \neq 0$. The real and imaginary parts of $z$ and $Z$ will be denoted $z = x + \mathrm{i}y$ resp. $Z = X + \mathrm{i}Y$ and the reciprocal function of $\mathsf{f}$ by $\mathsf{F}$: $z = \mathsf{F}(Z)$.

**a)** Check that $\mathsf{f}$ maps the circle $|z| = R_{\mathrm{J}}$ onto the line segment $|X| \leq 2R_{\mathrm{J}}$ in the complex $Z$-plane. (For purists: the reciprocal $\mathsf{F}$ is holomorphic on the complex $Z$-plane deprived of this line segment.)

**b)** Consider a uniform flow in the complex $Z$-plane with constant velocity field $V_0\,\vec{\mathrm{e}}_X$ parallel to the $X$-axis. On the line segment $|X| \leq 2R_{\mathrm{J}}$ lies a thin plate of length $4R_{\mathrm{J}}$, around which the fluid is flowing.

Since this flow is trivially incompressible and irrotational, give its complex velocity potential $\Phi(Z)$ and its streamlines.

**c)** Using the reciprocal mapping $\mathsf{F}$ — but *without* having to actually compute the form of $\mathsf{F}(Z)$! —, the uniform flow of question **i.b)** is mapped into the complex $z$-plane. What is the geometry of the flow in the $z$-plane? Give the associated complex potential $\phi(z)$ and the equation of the corresponding stream function. (*Hint*: you know $\Phi(Z)$ as a function of $Z$ and you know $Z = \mathsf{f}(z)$ — that's all you need!) Plot a few streamlines, using the value $R_{\mathrm{J}} = 1$ and assuming that $V_0$ is positive.

**ii.** Consider now the mapping

$$\mathsf{g}\colon \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \zeta = \mathsf{g}(z) \equiv z + c + \frac{(1+c)^2}{z+c} \tag{2}$$

where $c \in \mathbb{C}$.

**a)** Plot in the $\zeta$-plane the image by $\mathsf{g}$ of the unit circle $|z| = 1$ for $c = -0.17$ and $c = -0.12 + 0.1\mathrm{i}$.

**b)** In **i.c)** you found streamlines around the unit circle in the $z$-plane for a potential flow that tends to $V_0\,\vec{\mathrm{e}}_x$ in the limit $|z| \to \infty$. Can you deform this flow — and in particular its streamlines — to a potential flow in the $\zeta$-plane around the obstacle profiles you found in **ii.a)**?