Tutorial sheet 7

Discussion topic: What is a potential flow? What are the corresponding equations of motion?

13. Flow of a liquid in the vicinity of a gas bubble

We assume that the flow of the liquid is radial: $\vec{\mathsf{v}}(t,\vec{r}) = \mathsf{v}(t,r)\,\vec{\mathrm{e}}_r$, where the gas bubble is assumed to sit at $\vec{r} = \vec{0}$. Throughout the exercise, the effect of the liquid-gas surface tension—which gives rise to a difference in pressure between both sides of the liquid-gas interface—is neglected.

i. a) Show that the liquid's flow is irrotational. (*Hint*: one can avoid the computation of the curl!)

b) Assuming in addition that the flow is incompressible, derive the expression of $\mathsf{v}(t,r)$ in terms of the bubble radius $R(t)$ and its derivative $\dot{R}(t)$. Deduce therefrom the velocity potential.

ii. One assumes that the gas inside the bubble is an ideal gas which evolves adiabatically when the bubble radius varies, i.e. that its pressure—assumed to be uniform—and volume obey $\mathcal{P}\mathcal{V}^\gamma = \text{constant}$, where γ is the heat capacity ratio. Let $\mathcal{P}_0$ be the value of the pressure at infinity and R_0 the bubble radius when the gas pressure equals $\mathcal{P}_0$.

a) Neglecting the gas flow, give the expression of the pressure inside the bubble in terms of the radius.

b) Writing the Euler equation in terms of the velocity potential, show that $R(t)$ obeys the evolution equation

$$\ddot{R}(t)R(t) + \frac{3\big[\dot{R}(t)\big]^2}{2} = \frac{\mathcal{P}_0}{\rho}\left[\left(\frac{R_0}{R(t)}\right)^{3\gamma} - 1\right], \qquad (1)$$

where ρ is the liquid mass density.

iii. Suppose now that the bubble radius slightly oscillates about the equilibrium value R_0. Writing $R(t) = R_0[1+\epsilon(t)]$ with $|\epsilon(t)| \ll 1$, derive the (linear!) evolution equation for $\epsilon(t)$. What is the frequency f of such small oscillations?

Numerical application: calculate f for air ($\gamma = 1.4$) bubbles with $R_0 = 1$ mm and $R_0 = 5$ mm in water ($\rho = 10^3$ kg/m^3) for $\mathcal{P}_0 = 10^5$ Pa.

14. A characterization of potential incompressible flows

Let $\vec{\mathsf{v}}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r})$ denote a potential incompressible flow on a simply connected domain $\mathcal{V}$ of space, such that every point $\vec{r}$ of the boundary $\partial\mathcal{V}$ of the domain the normal component $\vec{\mathrm{e}}_n(\vec{r})\cdot\vec{\mathsf{v}}(t,\vec{r})$ takes a given (not necessary constant) value, where $\vec{\mathrm{e}}_n(\vec{r})$ is a unit vector normal to $\partial\mathcal{V}$. There can exist other incompressible flows on $\mathcal{V}$ obeying the same boundary condition, yet for instance with a different pressure field and with a nonzero vorticity: let $\vec{\mathsf{v}}'(t,\vec{r})$ denote the velocity field of such a flow. We want to show that the potential flow is the one that minimizes the total kinetic energy, i.e. that

$$E'_{\text{kin}} - E_{\text{kin}} \equiv \frac{\rho}{2}\int_{\mathcal{V}}\big[\vec{\mathsf{v}}'(t,\vec{r})\big]^2\,\mathrm{d}^3\vec{r} - \frac{\rho}{2}\int_{\mathcal{V}}\big[\vec{\mathsf{v}}(t,\vec{r})\big]^2\,\mathrm{d}^3\vec{r} \qquad (2)$$

is always positive. Note that in Eq. (2) we have used the incompressibility to pull the mass density ρ outside of the integrals.

i. Show that the difference $E'_{\text{kin}} - E_{\text{kin}}$ may be recast as the sum of two volume integrals, such that the integrand of the first one is a square—and thus always positive—while the integrand of the second one is the dot product $\vec{\mathsf{v}}(t,\vec{r})\cdot\big[\vec{\mathsf{v}}'(t,\vec{r}) - \vec{\mathsf{v}}(t,\vec{r})\big]$.

ii. Using the assumptions on the flows, show that $\displaystyle\int_{\mathcal{V}}\vec{\mathsf{v}}(t,\vec{r})\cdot\big[\vec{\mathsf{v}}'(t,\vec{r}) - \vec{\mathsf{v}}(t,\vec{r})\big]\,\mathrm{d}^3\vec{r} = 0$ and conclude.