Tutorial sheet 6

Throughout this exercise sheet we use Minkowski coordinates $\{x^\mu\}$ and set $c = 1$. For the sake of brevity, the x-dependence of fields will not be denoted.

11. Vorticity in a relativistic perfect fluid (part 2)

In this exercise, we consider a perfect fluid with a single conserved charge (referred to as "particle number"). Let n denote its local-rest-frame density.

i. Preamble: another form of the relativistic Euler equation

Let $h \equiv (e + \mathcal{P})/n$ be the enthalpy per particle. Show that you can rewrite the momentum-conservation equation for a perfect fluid in the form

$$n u^\mu \partial_\mu \big(h u_\nu\big) - n u_\nu u^\mu \partial_\mu h = -\partial_\nu \mathcal{P} - u_\nu u^\mu \partial_\mu \mathcal{P}. \tag{1}$$

(Cf. second equation in exercise **4.**).

iii. Dynamical vorticity

Let us define the (dynamical) *vorticity tensor* by the components

$$\Omega_{\mu\nu} \equiv \partial_\nu(h u_\mu) - \partial_\mu(h u_\nu). \tag{2}$$

a) Derive the identity

$$\Omega_{\mu\nu} = 2h\left[\omega_{\mu\nu} - \frac{1}{2}\big(a_\mu u_\nu - a_\nu u_\mu\big) + \frac{1}{2}\big(u_\mu \partial_\nu - u_\nu \partial_\mu\big)\ln h\right], \tag{3}$$

where the $\omega_{\mu\nu}$ are the components of the kinematic vorticity tensor defined in exercise **10.**.

b) Using the momentum-conservation equation (1), show

$$\Omega_{\mu\nu} u^\nu = \partial_\mu h - \frac{1}{n}\partial_\mu \mathcal{P}. \tag{4}$$

c) Define now a four-vector by $\Omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Omega_{\rho\sigma} u_\nu$. Why is it obviously orthogonal to the four-velocity? Show that it evolves according to

$$u^\nu \partial_\nu \Omega^\mu = 2 u^\nu \partial_\nu \big(h \omega^\mu\big), \tag{5}$$

where the vorticity 4-vector ω^μ was defined in exercise **10.**.

12. Two integral identities

Using the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, prove the following results for a system for which $T^{\mu\nu}$ vanishes outside a bound region of space.[1] In these identities, the roman indices take values from $i, j \in \{1, 2, 3\}$.

i. $\displaystyle \frac{\partial^2}{\partial t^2}\int T^{00} x^i x^j \, \mathrm{d}^3\vec{r} = 2\int T^{ij}\,\mathrm{d}^3\vec{r}$ — constituting the so-called *tensor virial theorem*.

ii. $\displaystyle \frac{\partial^2}{\partial t^2}\int T^{00} \big(x^i x_i\big)^2 \mathrm{d}^3\vec{r} = 4\int T^i{}_i\, x^j x_j\,\mathrm{d}^3\vec{r} + 8\int T^{ij} x_i x_j\,\mathrm{d}^3\vec{r}.$

[1] You already know $\displaystyle \frac{\partial}{\partial t}\int T^{0\mu}\,\mathrm{d}^3\vec{r} = 0$, expressing the conservation of energy and momentum.