Tutorial sheet 5

Throughout this exercise sheet we use Minkowski coordinates $\{x^\mu\}$ and set $c = 1$. For the sake of brevity, the x-dependence of fields will not be denoted. The projector orthogonal to the four-velocity (cf. exercise **2.**) is $\Delta^\mu{}_\nu \equiv u^\mu u_\nu + \delta^\mu_\nu$.

9. Equations of motions of a dissipative relativistic fluid

i. Let $\nabla^\mu \equiv \Delta^\mu{}_\nu \partial^\nu$ denote the gradient in the 3-space orthogonal to the 4-velocity (cf. exercise **2.**). Show the identity $\nabla_\mu u^\mu = \partial_\mu u^\mu$.

ii. Consider the general decomposition of the components of the energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (\mathcal{P} + \Pi)\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \varpi^{\mu\nu} \tag{1}$$

introduced in the lecture. Show that the resulting equations of motion can be expressed in the forms

$$u^\mu \partial_\mu \epsilon + (\epsilon + \mathcal{P} + \Pi)\nabla_\mu u^\mu + 2q_\mu a^\mu + \nabla^\mu q_\mu + \varpi^{\mu\nu} S_{\mu\nu} = 0 \tag{2}$$

(for the energy-conservation equation) and

$$\begin{aligned}(\epsilon + \mathcal{P} + \Pi)a_\mu + \nabla_\mu(\mathcal{P} + \Pi) + \nabla_\nu \varpi^\nu{}_\mu + a^\nu \varpi_{\mu\nu} - \big(S_{\nu\rho}\varpi^{\nu\rho}\big)u_\mu \\ + \Delta^\nu{}_\mu u^\rho \partial_\rho q_\nu + \left[\omega_{\mu\nu} + S_{\mu\nu} + \frac{4}{3}(\nabla_\rho u^\rho)\Delta_{\mu\nu}\right] q^\nu = 0\end{aligned} \tag{3}$$

(for the momentum-conservation equation), where we used the notations $a^\mu \equiv u^\nu \partial_\nu u^\mu$ (cf. exercise **7.**) and $S_{\mu\nu} \equiv \frac{1}{2}\Delta^\alpha{}_\mu \Delta^\beta{}_\nu\big(\partial_\beta u_\alpha + \partial_\alpha u_\beta\big) - \frac{1}{3}(\nabla_\rho u^\rho)\Delta_{\mu\nu}$ ("rate-of-shear tensor") while $\omega_{\mu\nu}$ is defined by Eq. (4) hereafter.

10. Vorticity in a relativistic perfect fluid (part 1)

Consider the *kinematic vorticity tensor* defined by its components

$$\omega_{\mu\nu} \equiv \frac{1}{2}\Delta^\alpha{}_\mu \Delta^\beta{}_\nu \big(\partial_\beta u_\alpha - \partial_\alpha u_\beta\big). \tag{4}$$

a) Why is no calculation necessary to prove the identity $\Delta^{\mu\nu}\omega_{\mu\nu} = 0$? Show the identity

$$\omega_{\mu\nu} = \frac{1}{2}\big(\partial_\nu u_\mu - \partial_\mu u_\nu + a_\mu u_\nu - a_\nu u_\mu\big), \tag{5}$$

where $a^\mu \equiv u^\nu \partial_\nu u^\mu$ (cf. exercise **7.**).

b) Define a four-vector by[1] $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\omega_{\rho\sigma}u_\nu$. What are its components in the local rest frame? What do you recognize?

c) (optional!) One can show that if $a^\mu = 0$, then the vorticity 4-vector obeys the evolution equation

$$u^\nu \partial_\nu \omega^\mu = -\frac{2}{3}(\partial_\rho u^\rho)\omega^\mu + S^\mu{}_\nu \omega^\nu, \tag{6}$$

where the rate-of-shear tensor $S^{\mu\nu}$ has been defined in exercise **9.**.

[1] The convention $\epsilon^{0123} = +1$ is used.