Tutorial sheet 4

Throughout this exercise sheet we use Minkowski coordinates $\{x^\mu\}$.

7. Non-relativistic limit

Let $u^\mu(\mathsf{x})$ denote the four-velocity of a fluid. With its help, one defines the four-vector field

$$a^\mu(\mathsf{x}) \equiv u^\nu(\mathsf{x})\partial_\nu u^\mu(\mathsf{x}). \tag{1}$$

How can one interpret a^μ? Compute the non-relativistic limit $|\vec{\mathsf{v}}| \ll 1 (= c!)$ of a^μ, where $\mathsf{v}^i = u^i/u^0$ is the 3-velocity associated with u^μ. What do you recognize? Is this consistent with your interpretation of a^μ?

8. A family of solutions of the dynamical equations for perfect relativistic fluids

Let $\tau^2 \equiv -x^\mu x_\mu$, where the "mostly plus" metric is used. Show that the following four-velocity, pressure and charge density constitute a solution of the equations describing the motion of a perfect relativistic fluid with equation of state $\mathcal{P} = K\varepsilon$ — where K is a constant — and a single conserved charge:

$$u^\mu(\mathsf{x}) = \frac{x^\mu}{\tau} \quad , \quad \mathcal{P}(\mathsf{x}) = \mathcal{P}_0\left(\frac{\tau_0}{\tau}\right)^{3(1+K)} \quad , \quad n(\mathsf{x}) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}\big(\sigma(\mathsf{x})\big), \tag{2}$$

with τ_0, $\mathcal{P}_0$, n_0 arbitrary constants and $\mathcal{N}$ an arbitrary function of a single argument, while σ is a function of spacetime coordinates with vanishing comoving derivative: $u^\mu(\mathsf{x})\partial_\mu\sigma(\mathsf{x}) = 0$.
Discuss the physical meaning of the solution (you may already know the limiting case with no conserved charge, $n = 0$).