Tutorial sheet 3

6. (1+1)-dimensional relativistic motion

Consider a (1+1)-dimensional relativistic motion along a direction denoted as z, where "1+1" stands for one time and one spatial dimension — throughout this exercise you may totally ignore the existence of the x and y directions. When the motion along the z-direction involves high velocities, it becomes appropriate to replace the Minkowski coordinates (t, z) by the *proper time* (of a comoving observer) τ and *spatial rapidity* ς such that[1]

$$\tau \equiv \sqrt{t^2 - z^2}, \quad \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \quad \text{where } |z| \leq t. \tag{1}$$

Throughout the exercise, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein's summation convention. For completeness, the relevant components of the metric tensor in Minkowski coordinates are $g_{tt} = -1$, $g_{zz} = +1$, $g_{tz} = g_{zt} = 0$.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau \cosh \varsigma, \quad z = \tau \sinh \varsigma. \tag{2}$$

(*Hint:* Recognize $\frac{1}{2} \log \frac{1+u}{1-u}$).
Deduce the following relationship between the basis vectors of the two coordinate systems

$$\begin{cases} \vec{\mathrm{e}}_\tau(\tau, \varsigma) = \cosh \varsigma \, \vec{\mathrm{e}}_t + \sinh \varsigma \, \vec{\mathrm{e}}_z \\ \vec{\mathrm{e}}_\varsigma(\tau, \varsigma) = \tau \sinh \varsigma \, \vec{\mathrm{e}}_t + \tau \cosh \varsigma \, \vec{\mathrm{e}}_z \end{cases} \tag{3}$$

and write down the metric tensor $g_{\tau\tau}$, $g_{\varsigma\varsigma} \ldots$ in the new coordinate system.
For the sake of completeness, give also the components $g^{\mu\nu}$ of the inverse metric tensor. Here and in the following, the unprimed indices μ, ν, ρ run over $\{\tau, \varsigma\}$.

ii. Inspiring yourself from what was done in the case of the two-dimensional Euclidean plane in the lecture, compute the Christoffel symbols $\Gamma^\mu_{\nu\rho}$.

iii. Let $N^\mu(\mathsf{x})$ denote the components of a 4-vector field. From now on, the x-dependence of the various fields (and/or their components) will be omitted for the sake of brevity.
Write down the covariant derivative $\mathrm{d}_\nu N^\mu \equiv \mathrm{d}N^\mu/\mathrm{d}x^\nu$ that generalizes to curvilinear (τ, ς) coordinates the derivative $\partial_{\nu'} N^{\mu'} \equiv \partial N^{\mu'}/\partial x^{\nu'}$ of Minkowski coordinates. Compute the 4-divergence $\mathrm{d}_\mu N^\mu$.

iv. Let $T^{\mu\nu}$ denote the components of a symmetric $\binom{2}{0}$-tensor field such that $T^{tz} = 0$.
Write down the covariant derivative $\mathrm{d}_\rho T^{\mu\nu}$ and compute $\mathrm{d}_\mu T^{\mu\nu}$ for $\nu \in \{\tau, \varsigma\}$.

v. Draw on a spacetime diagram—with t on the vertical axis and z on the horizontal axis—the lines of constant τ and those of constant ς.

[1] The coordinates (τ, ς) are sometimes called *Milne coordinates*.