Tutorial sheet 2

Discussion topic: What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal-energy density, pressure, and four-velocity? How is the latter defined?

Throughout this exercise sheet, we do not denote the x-dependence of the various fields (and/or their components) for the sake of brevity. In addition, a set of Minkowski components $\{x^\mu\}$ is used for x.

3. Energy-momentum tensor

Let $\mathcal{R}$ denote a fixed reference frame. Consider a perfect fluid whose local rest frame at a point x moves with velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}$. Show with the help of a Lorentz transformation that the Minkowski components of the energy-momentum tensor of the fluid at x are given to order $\mathcal{O}(|\vec{\mathsf{v}}|/c)$ by

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^1}{c} & (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^2}{c} & (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^3}{c} \\ (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^1}{c} & \mathcal{P} & 0 & 0 \\ (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^2}{c} & 0 & \mathcal{P} & 0 \\ (\epsilon+\mathcal{P})\dfrac{\mathsf{v}^3}{c} & 0 & 0 & \mathcal{P} \end{pmatrix}.$$

Check the compatibility of this result with the general formula for $T^{\mu\nu}$ given in the lecture.

4. Equations of motion of a perfect relativistic fluid

In this exercise, we set $c = 1$ for simplicity. Show that the energy-momentum conservation equation for a perfect fluid is equivalent to the two equations

$$u^\mu\partial_\mu\epsilon + (\epsilon+\mathcal{P})\partial_\mu u^\mu = 0 \quad \text{and} \quad (\epsilon+\mathcal{P})u^\mu\partial_\mu u^\nu + \nabla^\nu\mathcal{P} = 0, \tag{1}$$

where $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ (see exercise **2.**).

5. Isentropic flow

Consider a relativistic fluid with a single conserved charge. Let s resp. n denote the entropy density resp. the number density of the conserved charge, and u^μ the flow velocity.

Show that in an isentropic flow $[\partial_\mu(su^\mu) = 0]$ the "entropy per particle", defined as s/n, is conserved, i.e. $\mathrm{d}(s/n)/\mathrm{d}t = 0$.