Tutorial sheet 11

Discussion topic: Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects play a role?

For the sake of brevity, throughout this exercise sheet the dependence of the various fields on the space and time variables is not written.

20. Thermal convection between two vertical plates

Consider a fluid in a gravitational potential $-\vec{\nabla}\Phi = \vec{g} \equiv g\,\vec{e}_z$, contained between two infinite vertical plates at $x = \pm d/2$. When the plates have the same uniform temperature, there exist a static "isothermal" solution of the equations of motion describing the fluid, in which the latter is at the same temperature $T_{\rm eq}$ everywhere.

Assume that the plate at $x = -d/2$ resp. $x = +d/2$ is at a uniform temperature T_- resp. T_+ with $T_- < T_+$: this will induce a motion of the fluid, which we want to investigate. For simplicity, we shall assume that the motion is steady, and that it constitutes a small perturbation of the equilibrium state in which both temperatures are equal. Accordingly, the pressure, temperature and mass density are written in the form

$$\mathcal{P} = \mathcal{P}_{\rm eq} + \delta\mathcal{P}\ ,\ T = T_{\rm eq} + \delta T\ ,\ \rho = \rho_{\rm eq} + \delta\rho, \tag{1}$$

where the quantities with the subscript eq. refer to the equilibrium state, which need not be further specified.

i. Show that the relevant equations (VIII.8), (VIII.9), (VIII.12), (VIII.13) introduced in the lecture lead for the small quantities $\delta\mathcal{P}$, δT, $\delta\rho$ and $\vec{\mathsf{v}}$ to the system

$$\vec{\nabla}\cdot\vec{\mathsf{v}} = 0 \tag{2a}$$

$$\vec{\nabla}\big(\delta\mathcal{P}\big) = \delta\rho\,\vec{g} + \nu\rho_{\rm eq}\triangle\vec{\mathsf{v}} \tag{2b}$$

$$\vec{\mathsf{v}}\cdot\vec{\nabla}T_{\rm eq} = \alpha\triangle\big(\delta T\big) \tag{2c}$$

$$\delta\rho = -\alpha_{(\mathcal{V})}\rho_{\rm eq}\,\delta T \tag{2d}$$

where the stationarity assumption has already been used. How did you implement the assumed smallness of the "perturbations" to the static state? How can you already simplify Eq. (2c)?

ii. Let us assume that the new flow only depends on the x-coordinate, and that the y-direction plays no role at all; in particular, there is no component v_y. Let us further assume that the net mass flow through any plane $z =$ const. vanishes, i.e.

$$\int_{-d/2}^{d/2} \rho_{\rm eq}\,\mathsf{v}_z(x,y,z)\,{\rm d}x = 0 \tag{3}$$

for all y, z: this condition allows us to fully specify the "boundary" conditions to be obeyed by the velocity field.

a) Determine first the temperature-variation profile $\delta T(x)$ and deduce from it the mass density perturbation $\delta\rho(x)$. (*Hint*: Eqs. (2c)–(2d)).

b) Determine the velocity profile between the two plates. How do the streamlines look like?

iii. Time for some physics: what is absurd with the assumption of an infinite extent in the z-direction? Is there really heat convection in the flow determined in question **ii.**? Can you think of an (everyday-life) example—with finite plates!—corresponding to the setup considered here?

21. Speed of sound in ultrarelativistic matter

Consider a perfect fluid with the usual energy-momentum tensor. $T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (\epsilon+\mathcal{P})u^\mu u^\nu/c^2$. It is assumed that there is no conserved quantum number relevant for thermodynamics, so that the energy density in the local rest frame ϵ is function of a single thermodynamic variable, for instance $\epsilon = \epsilon(\mathcal{P})$. Throughout the exercise, Minkowski coordinates are used.

A background "flow" with uniform local-rest-frame energy density and pressure ϵ_0 and $\mathcal{P}_0$ is submitted to a small perturbation resulting in $\epsilon = \epsilon_0 + \delta\epsilon$, $\mathcal{P} = \mathcal{P}_0 + \delta\mathcal{P}$, and $\vec{\mathrm{v}} = \vec{0} + \delta\vec{\mathrm{v}}$.

i. Starting from the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, show that linearization to first order in the perturbations leads to the two equations of motion $\partial_t\delta\epsilon = -(\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\delta\vec{\mathrm{v}}$ and $(\epsilon_0 + \mathcal{P}_0)\partial_t\delta\vec{\mathrm{v}} = -c^2\vec{\nabla}\delta\mathcal{P}$.

ii. Show that the speed of sound is given by the expression $c_s^2 = \dfrac{c^2}{\mathrm{d}\epsilon/\mathrm{d}\mathcal{P}}$.

iii. Compute c_s for a fluid obeying the Stefan–Boltzmann law $\mathcal{P} = \dfrac{g\pi^2}{90}\dfrac{(k_\mathrm{B}T)^4}{(\hbar c)^3}$, with g the number of degrees of freedom (e.g. $g = 2$ for blackbody radiation).

Hint: You may find the Gibbs–Duhem relation useful...