Tutorial sheet 10

18. Propagation of internal waves in the ocean

The properties of several important instances of fluids found in nature—in particular their mass density ρ—depend on the altitude/depth z (oriented upwards): these fluids are said to be *stratified*. In the example of ocean water, ρ depends on depth "directly", i.e. because it is a function of pressure $\mathcal{P}$ which depends itself on z, but also "indirectly", inasmuch as depth influences the salinity [concentration in salt(s)], which in turn affects ρ.

The purpose of this exercise is to investigate internal waves in a stratified fluid at rest and in particular to exemplify a rather unusual dispersive behavior. Throughout, we consider a two-dimensional problem. Quantities related to the fluid in absence of wave are denoted with a subscript 0.

i. Brunt–Väisälä frequency

If a fluid particle is displaced vertically from its equilibrium position z_0 by an amount δz quickly enough, it will evolve adiabatically and without adjusting its salinity, so that when it is at $z_0 + \delta z$, its mass density ρ' differs from the equilibrium mass density $\rho_0(z_0 + \delta z)$ at that depth.

a) Considering the forces acting on the displaced fluid particle, show that Newton's second law gives

$$\rho' \frac{\mathrm{d}^2 \delta z}{\mathrm{d}t^2} = -g\big[\rho' - \rho_0(z_0 + \delta z)\big], \tag{1}$$

with g the acceleration due to gravity.
The *Boussinesq approximation*, which will also be used in question **ii.**, consists in approximating the mass density in the inertial term [left hand side of Eq. (1)] by the equilibrium value $\rho_0(z_0)$, while still keeping the "exact" value (here ρ') in the force term.

b) For the right-hand side of Eq. (1), one introduces a "potential density"[1] $\bar{\rho}$—which equals the mass density ρ under the same conditions of temperature and salinity, yet at a fixed reference pressure—such that the difference in the term between square brackets can be recast as $-(\mathrm{d}\bar{\rho}/\mathrm{d}z)\delta z$.

Under which condition on the derivative $\mathrm{d}\bar{\rho}/\mathrm{d}z$ is the equilibrium of the stratified fluid stable? In that case, what is the motion of the fluid particle? You may find it interesting to introduce the *Brunt–Väisälä "frequency"* defined by the relation $\omega_{\mathrm{B\text{–}V}}^2 \equiv -(g/\rho_0)\,\mathrm{d}\bar{\rho}/\mathrm{d}z$.

ii. Propagation of internal waves

Starting from a state of (stratified) rest, one considers small perturbations $\delta\rho(t,x,z)$, $\delta\mathcal{P}(t,x,z)$, $\delta\vec{\mathsf{v}}(t,x,z)$, assuming that the resulting flow is incompressible. After some uncomplicated, but also not very enlightening calculations, one finds that the (kinematic) incompressibility condition and the Euler equation in the Boussinesq approximation introduced above lead to the differential equation

$$\frac{\partial^4 \delta\mathsf{v}_z}{\partial t^2\,\partial x^2} + \frac{\partial^4 \delta\mathsf{v}_z}{\partial t^2\,\partial z^2} = -\omega_{\mathrm{B\text{–}V}}^2 \frac{\partial^2 \delta\mathsf{v}_z}{\partial x^2} \tag{2}$$

for the vertical component of $\delta\vec{\mathsf{v}}(t,x,z)$.

a) Show that the harmonic ansatz $\delta\mathsf{v}_z = \widetilde{\delta\mathsf{v}_z}\,\mathrm{e}^{-\mathrm{i}(\omega t - \vec{k}\cdot\vec{r})}$ leads to the dispersion relation

$$\omega = \pm\omega_{\mathrm{B\text{–}V}} \sin\beta_{\vec{k}}, \tag{3}$$

where $\beta_{\vec{k}}$ is the angle between the wave vector $\vec{k}$ and the z-direction. What do you find surprising here? Discuss the physics for various values of ω (consider 4 cases!).

[1] For more information, see `https://en.wikipedia.org/wiki/Potential_density`.

b) Compute the phase velocity $\vec{c}_\varphi(\vec{k})$ (remember that it is directed along $\vec{k}$) and the group velocity $\vec{c}_g(\vec{k}) \equiv \mathrm{d}\omega/\mathrm{d}\vec{k}$ following from the dispersion relation (3), and compare them with each other. Your result begs for comments!

19. A property of the solutions of the Korteweg–de Vries equation

In the lecture on surface waves on shallow water, it was shown that modulo a few approximations, the vertical displacement $\phi(\tau, \xi)$ of the sea surface obeys the Korteweg–de Vries equation

$$\frac{\partial \phi(\tau, \xi)}{\partial \tau} + 6\phi(\tau, \xi)\frac{\partial \phi(\tau, \xi)}{\partial \xi} + \frac{\partial^3 \phi(\tau, \xi)}{\partial \xi^3} = 0, \tag{4}$$

where τ resp. ξ is a (rescaled) time resp. position variable. Assuming that the wave is "localized", in the sense that ϕ and its derivatives vanish in the limit $|\xi| \to \infty$, show that the quantities

$$\int_{-\infty}^{\infty} \phi(\tau, \xi)\, \mathrm{d}\xi \quad \text{and} \quad \int_{-\infty}^{\infty} \big[\phi(\tau, \xi)\big]^2\, \mathrm{d}\xi \tag{5}$$

are conserved. What is the physical interpretation of these conservation laws?