Tutorial sheet 1

Discussion topic: What are the fundamental equations of the dynamics of a relativistic fluid?

Throughout this exercise sheet, we set $c = 1$, while the metric tensor has signature $(-,+,+,+)$.

1. Quantum number conservation

Consider a 4-current with components $N^\mu(x)$ obeying the continuity equation $\partial_\mu N^\mu(\mathsf{x}) = 0$, where the $\{x^\mu\}$ are Minkowski coordinates. Show that the quantity $\mathcal{N} = \int N^0(\mathsf{x})\,\mathrm{d}^3\vec{r}$ is a Lorentz scalar, by convincing yourself first that $\mathcal{N}$ can be rewritten in the form

$$\mathcal{N} = \int_{x^0=\text{const.}} N^\mu(\mathsf{x})\,\mathrm{d}^3\sigma_\mu, \tag{1}$$

where $\mathrm{d}^3\sigma_\mu = \dfrac{1}{6}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ is a 4-vector, with $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ the antisymmetric 4-tensor defined by

$$\mathrm{d}^3\mathcal{V}^{012} = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2, \quad \mathrm{d}^3\mathcal{V}^{021} = -\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^1, \quad \text{etc.}$$

and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric Levi–Civita tensor with the convention $\epsilon_{0123} = +1$, such that $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ represents the 3-dimensional hypersurface element in Minkowski space.

Hint: Take some time to discuss the orientation in space-time of $\mathrm{d}^3\sigma_\mu$ and to relate (as was done in the lecture on April 22) the "hypersurface integral" of Eq. (1) to your knowledge on surface integrals in Euclidean space.

2. A useful projector

In the lecture, the four-velocity of a fluid u^μ is defined as a timelike four-vector such that $u_\mu u^\mu = -1$.[1] Show that the tensor with components $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ defines a projector on the subspace orthogonal to the 4-velocity.

Denoting by d_μ the components of the (covariant) 4-gradient, we define $\nabla^\nu \equiv \Delta^{\mu\nu}\mathrm{d}_\mu$. Can you see the rationale behind this notation?

[1] Note that this also holds for the four-velocity of a point particle!