Solution to sheet 9

Discussion topic: Kinetic theory: which object does it consider? How does the fundamental equation look like? Which assumptions on the system underlie the description?

Throughout the exercise sheet, a system of units such that the constants c, $\hbar$ and k_B all equal 1 is used.

18. A possible scaling of the anisotropic flow coefficients of composite particles

Consider heavy-ion collisions resulting in the emission of protons and neutrons with the same transverse-momentum dependent anisotropic flow coefficients $v_{2,N}(p_T)$, ..., where the subscript N stands for nucleons. If a proton and neutron fly together — that is, with same transverse velocity or equivalently momentum — along the same direction for long enough, they may bind together into a deuteron (^{2}H). If three nucleons fly together, they may form a ^{3}H or ^{3}He nucleus.

Assuming that these light nuclei really result from the "coalescence" or 2 resp. 3 nucleons with the same transverse momentum, can you express their respective elliptic flow coefficients $v_{2,^2\mathrm{H}}(p_T)$, $v_{2,^3\mathrm{H}}(p_T)$, $v_{2,^3\mathrm{He}}(p_T)$ as a function of the elliptic flow $v_{2,N}$ of nucleons at an appropriate transverse momentum? Give $v_{2,^2\mathrm{H}}(p_T)$, resp. $v_{2,^3\mathrm{H}}(p_T)$ and $v_{2,^3\mathrm{He}}(p_T)$ up to second resp. third order in $v_{2,N}$.

Hint: You may want to go back to the level of the transverse momentum ($\mathbf{p}_T$) distributions.

Solution:

Let us start by writing the probability of finding one nucleon at a given $\vec{p_T}$. If you go back a few exercises ago you may find a similar expression to what follows

$$P_N(p_T) = \frac{1}{2\pi}(1 + 2v_{2,N}(p_T)\cos(2(\phi - \Psi_R))),$$

where we have assumed that there is only elliptic flow and that there are no correlations between particles. Notice that the probability also depends on the azimuth but I just write the p_T dependence.

In order to form a deuteron (^2H) we need to have two nucleons flying more or less together such that a bound state can be formed. This means that the azimuth of these nucleons must be the same (or pretty similar).

Therefore we compute the product of probabilities as

$$P_N(p_T)P_N(p_T) = \frac{1}{4\pi^2}(1 + 4v_{2,N}(p_T)\cos(2(\phi - \Psi_R)) + 4v_{2,N}^2(p_T)\cos^2(2(\phi - \Psi_R)))$$

where the result is no longer a probability since it is not normalized to unity (we could normalize it) but we can understand it as a distribution i.e. the non-normalized probability of finding a particle with momentum p_T and angle ϕ and another one with the same momentum. Notice that we use that both particles must have the same momentum since otherwise no bound state could be created and it also simplifies the calculations.

The elliptic flow coefficients is

$$v_2(p_T) = \frac{< P(p_T)\cos(2(\phi - \psi_R)) >}{< P(p_T) >},$$

where the average means an integral over the azimuth. Here you can see that it does not matter if you normalize your distribution or not since the normalization factor would vanish when computing the elliptic flow coefficient.

The above distribution for deuteron gives

$$v_{2,{}^2H}(2p_T) = \frac{4\pi v_{2,N}(p_T)}{2\pi + 4\pi v_{2,N}(p_T)} = \frac{2v_{2,N}(p_T)}{1 + 2v_{2,N}^2(p_T)} \approx 2v_{2,N}(p_T) = 0.19\,.$$

First of all the deuteron momentum is twice the nucleon one because of momentum conservation. Secondly notice that the elliptic flow of the deuteron is basically twice the one of the nucleons however you have to consider the shit in momentum. Last but not least, during the exercise we will use $v_{2,N}(p_T) = 0.1$ as a typical value in order to see an example.

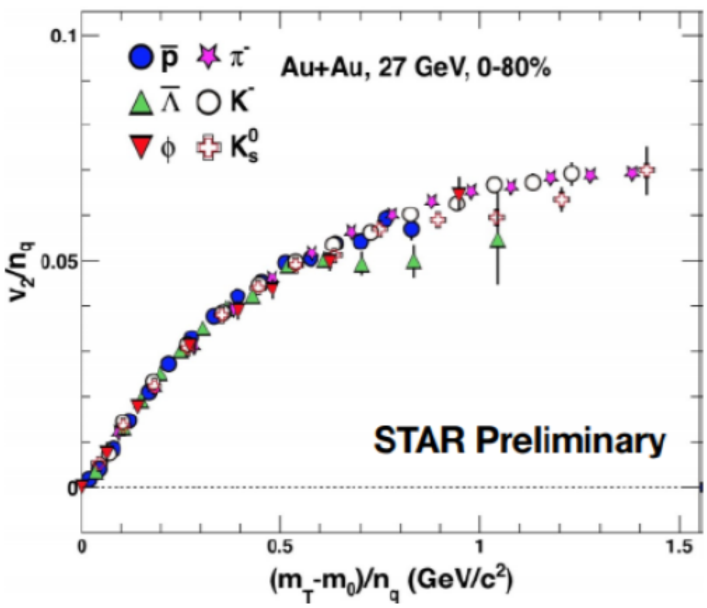
You can already guess how the game works if we want to form a tritium or a helium-3 which will have the same elliptic flow since both have three nucleons. The distribution in this case will read as

$$P_N^3(p_T) = 1 + 6v_{2,N}(p_T)\cos(2(\phi - \psi_R)) + 12v_{2,N}^2(p_T)\cos^2(2(\phi - \psi_R)) + 8v_{2,N}^3(p_T)\cos^3(2(\phi - \psi_R)),$$

Similarly to the case above the elliptic flow of the three-nucleons particles will be

$$v_{2,{}^3H/{}^3He}(3p_T) = \frac{3v_{2,N}(p_T) + 3v_{2,N}^3(p_T)}{1 + 6v_{2,N}^2(p_T)} \approx 3v_{2,N}(p_T) = 0.29.$$

This model is called coalescence model since it is based on grouping quarks into groups of two or three (for the case of quarks) to form either mesons or baryons which actually explains part of the measured $v_2(p_T)$ experimental results. We used nucleons instead of quarks but the idea is exactly the same.



On the figure you can see (more or less) that coalescence does actually work. On the vertical axis you have the v_2 divided by the number of quarks and on the horizontal axis something similar to p_T.

Actually you have $m_T - m_o = \sqrt{m^2 + p_T^2 - m^2} \approx p_T$ if the mass is negligible. At low momentum coalescence does not work but instead one finds mass scaling (showed on the lecture slides).

19. Thermodynamics of a relativistic gas of massive particles

Consider a gas of particles of mass m with a phase-space distribution given by the Maxwell–Jüttner distribution[1]

$$f(\mathsf{x}, \mathsf{p}) = C\, e^{p^\mu u_\mu / T}, \tag{1}$$

where C is a normalization constant while both u_μ and T are independent of x.

i. Compute the corresponding energy-momentum tensor, i.e. (why?) the energy density ε and pressure $\mathcal{P}$ of the gas, as a function of T.

You may find that your results look nicer when expressed in terms of modified Bessel functions of the second kind (of integer order n)

$$K_n(z) = \int_0^\infty e^{-z\cosh\lambda} \cosh(n\lambda)\, d\lambda,$$

where $\mathrm{Re}(z) > 0$.

Hint: Work in the appropriate reference frame!

Solution:

The energy momentum tensor can be computed from the particle distribution function as

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 E} p^\mu p^\nu f(x, p).$$

As you can see the energy momentum tensor does not depend on $\vec{p}$ and so it has to be integrated. Since our gas of particles is isotropic we know that the pressure will be the same in all directions.

We start by computing the pressure which can be written as

$$P_i = T^{ii} = \int \frac{d^3p}{(2\pi)^3 E} p_i^2 f(x, p)$$

We can solve that integral by using spherical coordinates and a change of variable as follows,

$$\int \frac{d^3p}{(2\pi)^3 E} p_i^2 f(x, p) = \frac{4\pi C}{3} \int \frac{dp}{(2\pi)^3 E} p^4 e^{-E/T}$$

We introduce the change of variables (motivated by the hint given)

$$\sqrt{p^2 + m^2} = e = m\cosh\alpha \qquad p = m\sinh\alpha.$$

Hence the pressure reads as

$$P_i = \frac{4\pi C}{3(2\pi)^3} \int m^4 \sinh^4(\alpha) e^{-m\cosh(\alpha)/T} d\alpha = \frac{\pi C}{6(2\pi)^3} m^4 (-4K_2(\frac{m}{T}) + K_4(\frac{m}{T}) + 3K_0(\frac{m}{T}))$$

where we have used that $\sinh^4(x) = \frac{1}{8}(-4\cosh(2x) + \cosh(4x) + 3)$.

[1] Beware! in the exponent the mostly-plus metric is used.

The result we have obtained is correct (hopefully) but it can simplified using the properties of Bessel functions. For example, a useful property is $K_n(z) = K_{n-2}(z) + \frac{2(n-1)}{z} K_{n-1}(z)$. Using these properties we find

$$P = P_x = P_y = P_z = \frac{C}{2\pi^2} m^2 T^2 K_2(\frac{m}{T}),$$

which has the proper dimensions (try to check it).

The energy density reads similarly

$$\epsilon = \int \frac{d^3p}{(2\pi)^3 E} E^2 f(x,p) = \frac{4\pi C}{(2\pi)^3} \int m^4 \sinh^2(\alpha) \cosh^2(\alpha) e^{-m\cosh(\alpha)/T} d\alpha = ...$$

$$= \frac{3C}{2\pi^2} m^2 T^2 K_2(\frac{m}{T}) + \frac{C}{2\pi^2} m^3 T K_1(\frac{m}{T})$$

Another option for computing these integrals is to change to the transverse mass and the rapidity. This might look more natural to us but then one has to integrate twice

$$\epsilon = \int \frac{d^3p}{(2\pi)^3 E} E^2 f(x,p) = \frac{C}{(2\pi)^2} \int dy\, dm_T\, d\phi\, m_T^3 \cosh^2 y\, e^{-m_T \cosh y/T}.$$

What about the other components of $T^{\mu\nu}$? They are all zero due to symmetry arguments, notice that they are integrals of odd powers of p_i.

In the small mass limit we can expand the Bessel functions as

$$K_n(z) = \frac{n!}{z^n}.$$

Therefore we find,

$$P = \frac{C}{\pi^2} T^4 \qquad \epsilon = \frac{3C}{\pi^2} T^4,$$

which gives our well known relation $\epsilon = 3P$. You have now found the e.o.s. of a massless gas!

Overall we have $T^{\mu\mu} = (\epsilon, P, P, P)$ and all the other components being zero. The energy momentum tensor becomes traceless in the massless limit.

You can now repeat the calculations using a BE or FD distribution.

ii. Compute the particle number density $n(T)$. How is it related to the pressure? What do you recognize?

Remark: this calculation of the particle number density actually *determines* the value of C.

Solution:

$$n = \int \frac{d^3p}{(2\pi)^3 E} E f(x,p) = ... = \frac{C}{2\pi^2} m^2 T K_2(\frac{m}{T}) = \frac{P}{T}.$$

If you now integrate over the volume (3dim. volume) you will obtain the number of particles N which will determine the constant C. Notice also that the particle density scales as T^3 contrary to the one power more we found for the energy density and pressure.

iii. Using your results for $\varepsilon(T)$ and $n(T)$, compute the energy per particle $\varepsilon(T)/n(T)$ in the non-relativistic limit $T \ll m$ up to order T^2. Interpret the first two terms of your result.

Hint: You may use the asymptotic expansion (valid for $|\arg z| < \frac{3\pi}{2}$)

$$K_n(z) \underset{z\to\infty}{\sim} \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\left[1 + \frac{4n^2-1}{8z} + \frac{(4n^2-1)(4n^2-9)}{2!(8z)^2} + \mathcal{O}(z^{-3})\right].$$

Solution: As suggested we expand the energy per particle for the non-relativistic limit.

$$\frac{\epsilon}{n} = 3T + m\frac{K_1(\frac{m}{T})}{K_2(\frac{m}{T})} \approx 3T - m - \frac{3}{2}T + \frac{15}{8}\frac{T^2}{m^2} = \frac{3}{2}T + m + \frac{15}{8}\frac{T^2}{m^2} \approx \frac{3}{2}T + m.$$

First, in the massless limit we have $\epsilon/n = 3T$. In the "classic" limit we find at first approximation $\epsilon/n = \frac{3}{2}T$ which looks just as the equipartition theorem (1/2 for each dimension basically). If you go further you get a correction factor which is the mass, as $\epsilon/n = \frac{3}{2}T + m$ which, perhaps, tells us that not all the energy of the system can be expended as thermal but part of it has to go to the mass or in other words, of the total energy E and amount N times the mass has to be extracted in order to create the particles and the rest can be used to increase the temperature from 0 to T. Further terms might be corrections to all of the above.

Remark: The Maxwell–Jüttner distribution is for “classical” particles and ignores both Bose–Einstein enhancement and Pauli’s exclusion principle.