Tutorial sheet 8

Throughout the exercise sheet, a system of units such that the constants c, $\hbar$ and k_B all equal 1 is used.

16. A toy equation of state

A long-used oversimplified Ansatz for the system created in high-energy nuclear collisions is that it would consist either of an ideal gas of massless pions or of an ideal gas of quarks and gluons with the respective pressures

$$\mathcal{P}_\pi(T) = \frac{g_\pi \pi^2}{90} T^4 \quad , \quad \mathcal{P}_{qg}(T) = \frac{g_{qg}\pi^2}{90} T^4 - B, \tag{1}$$

where the possible chemical potentials are ignored, with the corresponding energy densities given by

$$\epsilon(T) = T \frac{\partial \mathcal{P}(T)}{\partial T} - \mathcal{P}(T). \tag{2}$$

In Eq. (1), the positive "bag constant" B is a temperature-independent pressure exerted by the exterior of the quark–gluon system, keeping the latter confined to a small spatial region.[1]

i. Compute the energy and entropy density for either system.

ii. At a given temperature, the stable thermodynamic state is that with the larger pressure, corresponding to the smaller thermodynamic potential.[2]

a) Compute the temperature $T_{\rm tr}$ at which both pressures $\mathcal{P}_\pi$, $\mathcal{P}_{qg}$ are equal as a function of g_π, g_{qg} and B. Sketch the plot of the pressure of the stable phase as a function of T.

b) At $T_{\rm tr}$ a first-order phase transition takes place, at which both phases can coexist. Denoting by ξ the relative fraction of pion gas in the stable phase—$\xi = 1$ for $T < T_{\rm tr}$, $0 \leq \xi \leq 1$ for $T = T_{\rm tr}$, and $\xi = 0$ for $T > T_{\rm tr}$—, the energy density reads $\epsilon = \xi\epsilon_\pi + (1-\xi)\epsilon_{qg}$. Plot (sketch!) ϵ as a function of T, then $\mathcal{P}$ vs. ϵ, and eventually the speed of sound vs. ϵ.

c) Plot now the entropy density s as a function of temperature, again for the stable phase only. What do you notice?

In a first-order phase transition from a phase I to a phase II at a temperature $T_{\rm tr}$, one introduces the *latent heat* (per unit volume) $L_{\rm I\to II} \equiv T_{\rm tr}(s_{\rm II} - s_{\rm I})$. This represents the amount of energy that must be provided to the system to turn one unit volume of phase I into a unit volume of phase II.[3] Considering that the pion gas plays the role of phase I, compute the latent heat for the transition to the quark–gluon phase.

iii. In Eq. (1), g_π and g_{qg} are the degeneracy factors. Knowing that a fermionic degree of freedom only gives a contribution $\frac{7}{8}$ to this factor, explain why the values $g_\pi = 3$ and $g_{qg} = 37$ are used, the latter for a system with 2 massless quark flavors (u, d).

Taking $B = 0.4$ GeV/fm^3, compute the numerical values of $T_{\rm tr}$ and the latent heat per unit volume.

17. Free-streaming gas of particles

i. Let $f(t, \vec{r}, \vec{p})$ denote the single-particle phase-space distribution of a system of non-interacting, non-decaying particles evolving in the absence of long-range interactions deriving from a vector potential—yet possibly in the presence of space-dependent forces $\vec{F}(\vec{r})$ deriving from a constant scalar potential.

[1] "Outside" the quark–gluon system lies the QCD vacuum: B was originally introduced to model the confinement of massless quarks inside hadrons ("MIT bag model").

[2] The grand canonical potential of a system with volume $\mathcal{V}$ is $\Omega = -\mathcal{P}\mathcal{V}$.

[3] Or equivalently, the amount of energy released when one unit volume of phase II turns into phase I.

Consider the particles which are at time t are in an infinitesimal phase-space volume $\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}$ about point $(\vec{r},\vec{p})$. Where are these particles at the instant $t+\delta t$? Show that the volume element $\mathrm{d}^3\vec{r}'\,\mathrm{d}^3\vec{p}'$ they then occupy equals $\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}$. Derive the partial differential equation governing the evolution of f.

Hint: Particle number is conserved, and Newton's second law holds.

ii. Free-streaming solution

In the absence of collisions and of external forces, the (collisionless!) relativistic Boltzmann equation governing the evolution of the (on-shell) single-particle phase-space distribution $f(\mathsf{x},\vec{p})$ reads

$$p^\mu \partial_\mu f(\mathsf{x},\vec{p}) = 0 \tag{3}$$

where the p^μ are the components of the four-momentum p.

Check that if the phase-space distribution equals a function $F_i(\vec{r},\vec{p})$ at some initial instant t_i (in a given reference frame), then at a later time t the distribution is given by

$$f(t,\vec{r},\vec{p}) = F_i\bigg(\vec{r} - \frac{\vec{p}}{p^0}(t-t_i), \vec{p}\bigg). \tag{4}$$