Solution to sheet 8

Throughout the exercise sheet, a system of units such that the constants c, $\hbar$ and k_B all equal 1 is used.

16. A toy equation of state

A long-used oversimplified Ansatz for the system created in high-energy nuclear collisions is that it would consist either of an ideal gas of massless pions or of an ideal gas of quarks and gluons with the respective pressures

$$\mathcal{P}_\pi(T) = \frac{g_\pi \pi^2}{90} T^4 \qquad , \qquad \mathcal{P}_{qg}(T) = \frac{g_{qg}\pi^2}{90} T^4 - B, \tag{1}$$

where the possible chemical potentials are ignored, with the corresponding energy densities given by

$$\epsilon(T) = T \frac{\partial \mathcal{P}(T)}{\partial T} - \mathcal{P}(T). \tag{2}$$

In Eq. (1), the positive "bag constant" B is a temperature-independent pressure exerted by the exterior of the quark–gluon system, keeping the latter confined to a small spatial region.[1]

i. Compute the energy and entropy density for either system.

Solution:
The energy density follows straight from the above equations. The results are:

$$\epsilon_\pi = \frac{3g_\pi\pi^2}{90} T^4 = 3P_\pi \qquad \epsilon_{qg} = \frac{3g_{qg}\pi^2}{90} T^4 + B = 3P_{qg} + 4B$$

For the entropy you can remember that we derived on sheet 7 that $d\epsilon = Tds$ in the absence of conserved charges. Since we know the explicit form of the energy density we can solve that equation which leads[2]. More easily you can use $e + p = Ts$.

$$s_\pi = \frac{4g_\pi\pi^2}{90} T^3 \qquad s_{qg} = \frac{4g_{qg}\pi^2}{90} T^3$$

where the two entropies are quite different due to the degeneracy factors. Also notice that B has disappeared. Both entropies vanish in the limit T=0 as expected by the third law of thermodynamics.

ii. At a given temperature, the stable thermodynamic state is that with the larger pressure, corresponding to the smaller thermodynamic potential.[3]

a) Compute the temperature T_{tr} at which both pressures $\mathcal{P}_\pi$, $\mathcal{P}_{qg}$ are equal as a function of g_π, g_{qg} and B. Sketch the plot of the pressure of the stable phase as a function of T.

Solution: The temperature at which the pressures are equivalent can be obtained from eq.(1) which leads to

$$T_{tr} = \left(\frac{90B}{\pi^2(g_{qg} - g_\pi)} \right)^{1/4},$$

where we see that it only has a solution if $g_{qg} > g_\pi$.

[1] "Outside" the quark–gluon system lies the QCD vacuum: B was originally introduced to model the confinement of massless quarks inside hadrons ("MIT bag model").

[2] To solve it just use $s = \int \frac{d\epsilon}{T} dT$.

[3] The grand canonical potential of a system with volume $\mathcal{V}$ is $\Omega = -\mathcal{P}\mathcal{V}$.

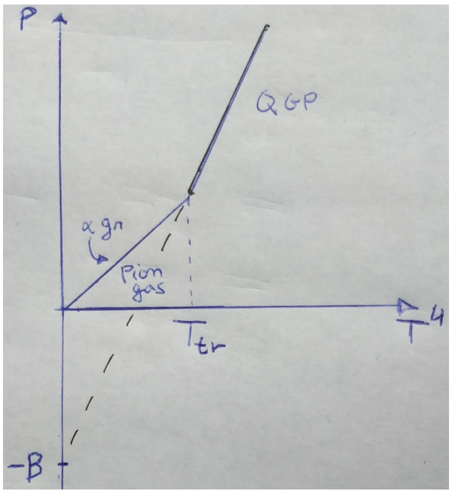

Figure 1: Pressure of the stable phase as function of the temperature. The QGP phase dominates at large temperatures while at lower ones the pion gas is the stable.

On the sketch we can see that at low temperatures the pressure scales linearly with T^4 and proportional to the pion degeneracy factor meanwhile in the QGP phase which occurs at high temperatures the pressure scales faster. This applies to most of the sketches we will see.

b) At $T_{\rm tr}$ a first-order phase transition takes place, at which both phases can coexist. Denoting by ξ the relative fraction of pion gas in the stable phase—$\xi = 1$ for $T < T_{\rm tr}$, $0 \leq \xi \leq 1$ for $T = T_{\rm tr}$, and $\xi = 0$ for $T > T_{\rm tr}$—, the energy density reads $\epsilon = \xi\epsilon_\pi + (1-\xi)\epsilon_{qg}$. Plot (sketch!) ϵ as a function of T, then $\mathcal{P}$ vs. ϵ, and eventually the speed of sound vs. ϵ.

Solution:

Again, at low temperature the stable solution is the pion gas and above the phase transition it is the qgp phase.

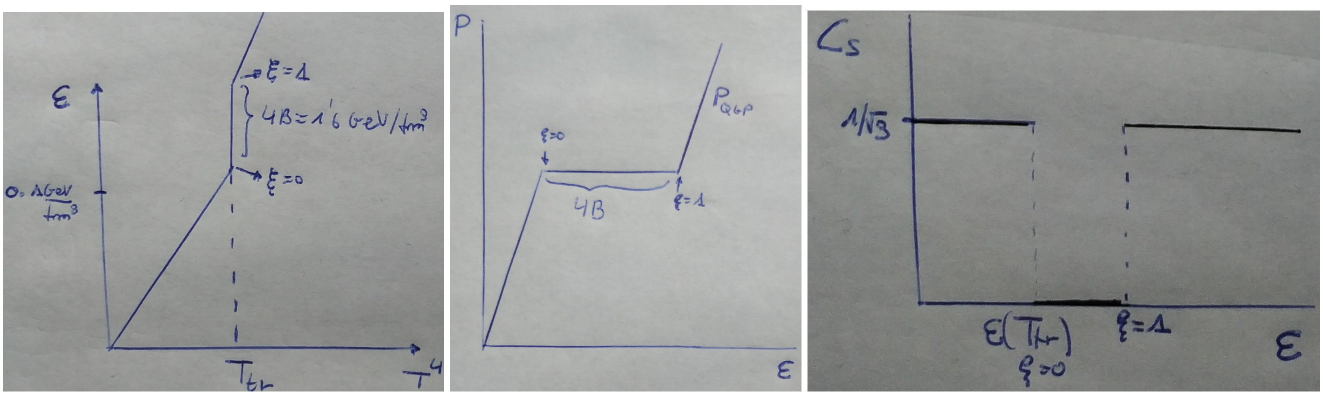

Figure 2: Sketeches of ϵ vs T, P vs T and C_s vs ϵ.

The energy densities at the phase transition are

$$\epsilon_\pi \approx 0.11 GeV/fm^3 \qquad \epsilon_{qg} \approx 1.70 GeV/fm^3$$

and notice that the volumes of the two phases do not need to be equal.

The pressure is constant at the phase transition since this is how we have defined the phase transition. The speed of sound is a bit more tricky, outside the phase transition it is a constant corresponding to the e.o.s ϵ=3P but during the phase transition it becomes zero because at the phase transition the pressure is a constant hence $c_s^2 \equiv \frac{\partial P}{\partial \epsilon} = 0$. This is an important result, the speed of sound is zero during the phase transition!

c) Plot now the entropy density s as a function of temperature, again for the stable phase only. What do you notice?

In a first-order phase transition from a phase I to a phase II at a temperature $T_{\rm tr}$, one introduces the *latent heat* (per unit volume) $L_{\rm I\to II} \equiv T_{\rm tr}(s_{\rm II} - s_{\rm I})$. This represents the amount of energy that must be provided to the system to turn one unit volume of phase I into a unit volume of phase II.[4] Considering that the pion gas plays the role of phase I, compute the latent heat for the transition to the quark–gluon phase.

Solution:

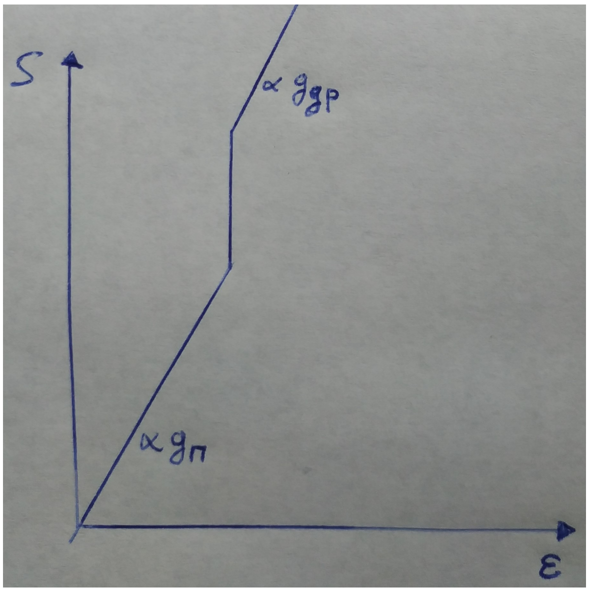

Figure 3: Sketch of the entropy vs T (there is a missprint on the "x" axis where it reads ϵ instead of T^3.) At the transition temperature the entropy changes drastically.

We can see that the entropy density rises with temperature proportional to the corresponding degeneracy factor. Also, at the phase transition it increases drastically (more than a factor 10).

The latent heat can be computed as

$$L_{\pi\to \rm qg} \equiv T_{\rm tr}(s_{\rm qg} - s_\pi) = \frac{4(g_{qg}-g_\pi)\pi^2}{90}\frac{90B}{\pi^2(g_{qg}-g_\pi)} = 4B = \epsilon_{qg}(T_{tr}) - \epsilon_\pi(T_{tr})$$

iii. In Eq. (1), g_π and g_{qg} are the degeneracy factors. Knowing that a fermionic degree of freedom only gives a contribution $\frac{7}{8}$ to this factor, explain why the values $g_\pi = 3$ and $g_{qg} = 37$ are used, the latter for a system with 2 massless quark flavors (u, d).

[4] Or equivalently, the amount of energy released when one unit volume of phase II turns into phase I.

Taking $B = 0.4$ GeV/fm^3, compute the numerical values of T_{tr} and the latent heat per unit volume.

Solution: The latent heat per unit volume is simply

$$L = 4B = 1.6GeV/fm^3.$$

The temperature is

$$T_{tr} = \left(\frac{90B}{\pi^2(g_{qg} - g_\pi)}\right)^{1/4} \approx 170MeV$$

Noitce that B has units of energy density but the result we got is just an energy. This is because, when using natural units we have that $\hbar c = 197.3$ MeV fm =1 (useful to remember).

Finally the degeneracy factors are obtained as follows:

For the pion gas g=3 since there are 3 pions (that was an easy one!).

For the qgp case it is more complicated. The degeneracy will be the sum of the bosonic d.o.f and the fermionic d.o.f. in the way $g = g_b + \frac{7}{8} g_f$. Gluons are bosons and have spin, therefore $g_b = 2\cdot(N_c^2 - 1) = 16$ since there are 3 colors. The factor 2 comes from the helicity or spin and the factor 8 due to gluons having 8 possible color states (octet). For the fermions we have two flavors each with its antiquark and again a factor 2 due to helicity or spin but this time each quark can have one color and there are three colors hence $g_f = 2 \cdot 2 \cdot 2 \cdot 3 = 24$. All in all we get $g_{qg} = 16 + \frac{7}{8} 24 = 37$.

17. Free-streaming gas of particles

i. Let $f(t, \vec{r}, \vec{p})$ denote the single-particle phase-space distribution of a system of non-interacting, non-decaying particles evolving in the absence of long-range interactions deriving from a vector potential— yet possibly in the presence of space-dependent forces $\vec{F}(\vec{r})$ deriving from a constant scalar potential. Consider the particles which are at time t are in an infinitesimal phase-space volume $\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}$ about point $(\vec{r}, \vec{p})$. Where are these particles at the instant $t + \delta t$? Show that the volume element $\mathrm{d}^3\vec{r}'\,\mathrm{d}^3\vec{p}'$ they then occupy equals $\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}$. Derive the partial differential equation governing the evolution of f.
Hint: Particle number is conserved, and Newton's second law holds.

Solution: A non rigorous solution is: Two observers must agree on the number of particles inside a given volume ($\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}\quad \mathrm{d}^3\vec{r}'\,\mathrm{d}^3\vec{p}'$). The number of particles will be $N = \int_{x,p} f(t, r, p) d^3x d^3p$. Since the expression holds for any integration volume that implies that

$$f(t, x, p)d^3rd^3p = f(t', x', p')d^3r'd^3p'.$$

The phase space volume is invariant under a Lorentz boost since one gets $d^3r' = d^3r/\gamma$ (length contraction) and momentum is instead $d^3p' = \gamma d^3p$. Therefore the product is invariant. If that is the case it means also that $f(t, x, p) = f(t', x', p')$ i.e. the densities are equal. You can find a better explanation at "The relativistic Boltzmann equation by Cario Cercignani and Gilberto Medeiros Kremer ".

If you want a more correct form you can use Liouville's theorem and Hamilton's equations to proof rigorously that " As the systems contained in a tiny region of phase space evolve according to classical mechanics, the volume they occupy remains constant. And because the volume is constant, the probability density remains constant as well".

At an instant $t + \delta t$ the density will read as $f(t + \delta t, x + \frac{dx}{dt}dt, p + \frac{dp}{dt}dt$.

Therefore, we can compute df/dt (how much the density changes with time) as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial p}{\partial t}\frac{\partial f}{\partial p} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + F\frac{\partial f}{\partial p} = 0.$$

It is equal to zero because we said that the density is constant. Notice that, if there were collisions on the system then this would not be true and we would write $= C$ which is the collision kernel. But, for our case,

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \mathbf{F}\frac{\partial f}{\partial \mathbf{p}} = 0.$$

ii. Free-streaming solution

In the absence of collisions and of external forces, the (collisionless!) relativistic Boltzmann equation governing the evolution of the (on-shell) single-particle phase-space distribution $f(\mathsf{x}, \vec{p})$ reads

$$p^\mu \partial_\mu f(\mathsf{x}, \vec{p}) = 0 \tag{3}$$

where the p^μ are the components of the four-momentum p.

Check that if the phase-space distribution equals a function $F_i(\vec{r}, \vec{p})$ at some initial instant t_i (in a given reference frame), then at a later time t the distribution is given by

$$f(t, \vec{r}, \vec{p}) = F_i\bigg(\vec{r} - \frac{\vec{p}}{p^0}(t - t_i), \vec{p}\bigg). \tag{4}$$

Solution: For a collisionless system and without external forces all that remains is free streaming. That means that the system still evolves because particles have a velocity and therefore they move. In that case

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} = 0$$

or equivalently (multiplying by the energy)

$$p^0\frac{\partial f}{\partial t} + p^0\mathbf{v}\frac{\partial f}{\partial \mathbf{x}} = p^\mu \partial_\mu f(x, p) = 0$$

Finally, if the distribution at some time t is F_i then at later times it will be

$$f(t, \vec{r}, \vec{p}) = F_i\bigg(\vec{r} - \frac{\vec{p}}{p^0}(t - t_i), \vec{p}\bigg).$$

Which follows easily since for free streaming momentum can not change and the position varies according to the velocity (no big secret).