# Tutorial sheet 7

**Discussion topic:** Relativistic fluid dynamics: what are the fundamental equations? Which further ingredients are needed to model a specific system? Which assumption(s) underlie the description?

## 14. Local thermodynamics

In your Statistical Physics lectures, thermodynamics and in particular the fundamental thermodynamic relations were formulated for extensive quantities, as e.g.

$$U = TS - \mathcal{P}\mathcal{V} + \sum_a \mu_a N_a \quad , \quad \mathrm{d}U = T\,\mathrm{d}S - \mathcal{P}\mathrm{d}\mathcal{V} + \sum_a \mu_a\,\mathrm{d}N_a$$

for a system with several conserved quantum numbers. The purpose of the present exercise is to derive the equivalent relations between the densities of the extensive thermodynamic parameters: energy density[1] $\epsilon$, entropy density $s$, number densities $n_a$.[2]

**i.** The relation for the internal energy $U$ translate at once in a relation for $\epsilon$. Using the latter and the relation for $\mathrm{d}U$, show that the differentials of energy density and pressure obey

$$\mathrm{d}\epsilon = T\,\mathrm{d}s + \sum_a \mu_a\,\mathrm{d}n_a \quad , \quad \mathrm{d}\mathcal{P} = s\mathrm{d}T + \sum_a n_a\,\mathrm{d}\mu_a \tag{1}$$

respectively. The second of these relations is the local version of the *Gibbs–Duhem relation*.

**ii.** Focusing on the case with no conserved quantum number, deduce from the previous relations an expression for the derivative

$$c_s^2 \equiv \frac{\partial \mathcal{P}}{\partial \epsilon} \tag{2}$$

in terms of the entropy density $s$ and the temperature $T$.

Assuming that $c_s^2$ is a constant, in which case Eq. (2) is equivalent to the simple "equation of state" $\mathcal{P} = c_s^2\epsilon$, show the scaling behaviors

$$\epsilon \propto T^{1+1/c_s^2} \quad , \quad s \propto T^{1/c_s^2}. \tag{3}$$

What do you find in the case $\epsilon = 3\mathcal{P}$? Since this is the equation of state for an ideal gas of ultrarelativistic particles, you may refresh your knowledge on the photon gas from your Statistical Physics lectures.

## 15. Relativistic fluid dynamics

The dynamics of a perfect relativistic fluid without conserved charge is entirely governed by the equation $\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu} = \epsilon u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu}$, with $u^\mu$ the 4-velocity and $\Delta^{\mu\nu} \equiv u^\mu u^\nu + g^{\mu\nu}$, where the mostly plus convention is used for the metric tensor (so that $u_\mu u^\mu = -1$).

Throughout the exercise, $c = 1$ and the x-dependence of fields is not written.

The fundamental equation of motion implies *automatically* entropy conservation, $\partial_\mu(su^\mu) = 0$. Using thermodynamic relations from question **i.** of the previous exercise, show that in a perfect relativistic fluid the temperature evolves according to the equation

$$u^\mu \partial_\mu (T u^\nu) = -\partial^\nu T. \tag{4}$$

Deduce from the latter and from a result from exercise **14.ii.** the equation

$$u^\mu \partial_\mu u^\nu = -c_s^2 \frac{\nabla^\nu s}{s}, \tag{5}$$

where $\nabla^\nu \equiv \Delta^{\nu\mu}\partial_\mu$. How can you interpret this equation?

[1] The equations take the same for in the non-relativistic and the relativistic cases.
[2] Note that these are the volume densities, not the specific densities used in some textbooks.